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EXACT RESULTS FOR WILSON LOOPS
IN
 $\mathcal{N} = 4$ SUPERSYMMETRIC YANG MILLS THEORY

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Chapter 1

Introduction

1.1 The $\mathcal{N} = 4$ SYM Theory

The $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory is a very intriguing quantum field theory. It has a rich mathematical structure and the magic of supersymmetry makes it more constrained and amenable to exact treatment. For these reasons it is probably the main candidate to be the first quantum field theory exactly solvable in four dimensions.

The theory has the maximal number of supercharges for a four dimensional theory without gravity (namely sixteen supercharges) and it shows very interesting features. The main one is certainly the fact that it is finite and exactly conformal at the quantum level. Perturbative arguments [1] and general considerations [2, 3, 4] suggest in fact that the β function of the theory could be zero to all orders. This peculiarity makes the theory simpler to treat but, at the same time, makes it very different from the ordinary quantum field theories like quantum chromodynamics.

The simplest way to construct the $\mathcal{N} = 4$ SYM is via dimensional reduction of the $\mathcal{N} = 1$ supersymmetric Yang-Mills in ten dimensions. Its field content comprises a gluon A_μ ($\mu = 0, 1, 2, 3$), six real scalars ϕ^I ($I = 4 \dots 9$) and four Weyl spinors which can be rearranged as a sixteen components ten-dimensional Majorana-Weyl spinor Ψ . In terms of these fields the Lagrangian reads

$$\mathcal{L}_{\mathcal{N}=4} = \text{Tr} \left[-\frac{1}{2}(F_{\mu\nu})^2 + (D_\mu \phi^I)^2 - \sum_{i < j} [\phi^I, \phi^J]^2 + i\bar{\Psi} \Gamma_\mu D_\mu \Psi - \bar{\Psi} \Gamma_I [\phi^I, \Psi] \right] \quad (1.1)$$

where covariant derivatives and the field strength are defined as usual: $D_\mu = \partial_\mu - i[A_\mu, \]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Here the Γ_I are the Dirac matrices in ten dimensions.

The global symmetries of the theory are :

- **Conformal Symmetry**, forming the group $SO(2,4)$ generated by translations P^μ , Lorentz transformations $L_{\mu\nu}$, Dilation D and Special Conformal transformations K_μ
- **R-Symmetry**, forming the group $SU(4)_R$ generated by T^a with $a = 1\dots 15$
- **Poincaré Supersymmetries**, generated by the Supercharges Q_α^a and $\bar{Q}_{\dot{\alpha}a}$ with $a = 1\dots 4$
- **Conformal Supersymmetries**, generated by the Superconformalcharges S_α^a and $\bar{S}_{\dot{\alpha}a}$ with $a = 1\dots 4$.

All these symmetries fit into the full supergroup $PSU(2,2|4)$ whose elements are schematically represented by the supermatrice of the form

$$\left(\begin{array}{c|c} P_\mu, L_{\mu\nu}, D, K_\mu & Q_\alpha^a, \bar{S}_{\dot{\alpha}a} \\ \hline S_\alpha^a, \bar{Q}_{\dot{\alpha}a} & T^a \end{array} \right).$$

Thanks to the AdS/CFT correspondence the $\mathcal{N} = 4$ SYM has been in focus of theoretical research for the past decade. This duality relates it to a type IIB superstring theory on a curved background [5]. Connections between gauge and string theories are not a novelty in theoretical physics, they have been a wide field of research since the 1974 when all the story has begun. In a seminal paper [6] 't Hooft showed that the perturbative expansion of gauge theories can be rearranged so that it displays a stringy behavior. He considered a $U(N)$ theory at large N while keeping $\lambda = g^2 N$ fixed. In this limit any Feynman diagram can be viewed as a triangulation of a closed oriented two dimensional Riemann surface. Quite surprisingly one finds that the sum over the Feynman graphs can be reinterpreted as the perturbative expansion of a closed string theory if we identify $1/N$ with the string coupling constant. However 't Hooft beautiful work did not give a precise prescription to find what string theory is dual to a particular gauge theory.

Other non-trivial examples of gauge/string correspondence are the relations between a zero dimensional field theory (namely a matrix model) and two dimensional quantum gravity (see [7] and reference therein) and the relation between QCD_2 and string theory that we will briefly analyze in the *appendix B* [8, 9, 10].

About fifteen years ago, in a beautiful paper [5] Maldacena proposed a conjecture that relates $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions to a superstring theory. Afterwards some important aspects of the correspondence were clarified in the two nice papers [11, 12] (for a comprehensive review on the subject see [13] and [14]). The precise form of the conjecture states that

$\mathcal{N} = 4$ Supersymmetric Yang Mills theory (SYM) in four dimensions with gauge group $SU(N)$ and coupling constant g_{YM}
\equiv
Type IIB Superstring Theory on $AdS_5 \times S^5$ with string coupling g_s AdS_5 and S^5 with the same radius R the self-dual RR-five form F_5 has an integer flux trough the five-sphere $\int_{S^5} F_5^+ = N$

with the identification

$$g_s = g_{YM}^2 \quad R^4 = 4\pi g_s N \alpha^2$$

and where the *VEV* of the axionic field of the type IIB string theory $\langle C \rangle$ is equal to the instantonic angle θ_I of the $\mathcal{N} = 4$ SYM. This duality is more general than 't Hooft original correspondence since it is expected to hold not only in the large N -limit but also for finite N and for all value of the coupling g_s .

One of the main features of the AdS/CFT correspondence is that it is an *ultra-violet/infrared duality*. Let us briefly review what it means. On the gauge side perturbation theory is valid if the coupling $\lambda = g_{YM}^2 N \ll 1$. In the dual picture this limit corresponds to the region $\frac{R}{l_s} \sim \lambda^{1/4} \ll 1$ (with $l_s = \sqrt{\alpha}$ the string length) and thus we are in a limit where the supergravity approximation is not reliable since we have to take into account the massive string states. Conversely, if on the string side the supergravity approximation is valid, the $\mathcal{N} = 4$ SYM is strongly coupled. In summary when a theory is in a weak couple regime (manageable with the usual perturbative techniques) its dual is in a strongly coupled one (difficult to treat). This peculiarity is interesting since in principle one can easily approach the gauge theory non-perturbatively by using the dual classical supergravity and apart from lattice techniques there is no other straight way to treat them. At the same time it makes the conjecture hard to prove since we do not know how to study string and gauge theories for generic value of the coupling constant.

The idea is thus to look at a particular limit of the conjecture in which the duality becomes more tractable. For example a weaker equivalence is obtained considering the 't Hooft limit that consists in keeping λ fixed and letting N to infinity. The large N limit of the $\mathcal{N} = 4$ SYM is thus expected to be dual to the classical type IIB string theory on $AdS_5 \times S^5$. The $1/N$ corrections in the gauge theory correspond on the string side to string loops contributions (in g_s). If instead we let N to infinity and set $\lambda \gg 1$ the correspondence is between the strongly coupled $\mathcal{N}=4$ SYM in the large N limit and the type IIB supergravity on $AdS_5 \times S^5$.

The first step towards testing the duality can be made in terms of symmetry. Both the $\mathcal{N} = 4$ SYM and the type IIB superstring theory have the same global symmetry, the superconformal $PSU(2, 2|4)$ group where, more precisely, the conformal

group $SO(2, 4)$ and the $SU(4)$ R -symmetry of the gauge theory correspond respectively to the AdS_5 and S^5 isometries. However to define better the correspondence and to prove it, it is necessary to establish a sort of dictionary between the two theories, a prescription for comparing physical quantities and amplitudes. Following the idea of the holographic principle [15, 16] that states that a physical content of a quantum gravity theory defined on a region M can be encoded by another theory that lives on the boundary ∂M , a concrete realization of the duality has been done independently in [11] and [12].

The $\mathcal{N} = 4$ SYM in four dimensions (CFT) is specified by a complete set of conformal operators $\mathcal{O}_h(x)$ and lives in Minkowski space-time (the boundary of AdS_5). It is dual to a five dimensional theory obtained by the Kaluza-Klein compactification of the type IIB string theory on $AdS_5 \times S^5$ over the five sphere. On the gravity side a field $h(x, x_5)$, the excitation of the string background coming from the S^5 compactification, knows about the operator $\mathcal{O}_h(x)$ in the CFT via the boundary coupling $\int d^4x \mathcal{O}_h(x) h(x, x_5)_{x_5=0}$; namely the restriction of the five dimensional field $h(x, x_5)$ to the boundary of AdS is the source that couples with the operator $\mathcal{O}_h(x)$ (a complete list of operators in $\mathcal{N} = 4$ SYM and of the corresponding fields in AdS is given in the table 7 of ref.[14]). A natural mapping between the correlators in the SYM theory and the dynamics of string theory is given as follows.

The generating functional $Z_{\mathcal{O}}[h(x, 0)] = \langle e^{\int d^4x \mathcal{O}_h(x) h(x, 0)} \rangle$ for the connected green functions of the operator $\mathcal{O}_h(x)$ is related to the partition function $Z_g[h(x, x_5)] = e^{-S[h(x, x_5)]}$ on the string side by the relation

$$Z_{\mathcal{O}}[h(x, 0)] = e^{-S[h(x, x_5)]}. \quad (1.2)$$

In the supergravity approximation, $S[h(x, x_5)]$ is just the type IIB supergravity action on AdS_5 while, beyond this limit, $S[h(x, x_5)]$ will have also to include α' corrections due to massive string effects.

We stress that to test the correspondence we have to look for protected or exactly solvable operators. Only in this case in fact, the comparison between the two sides of the correspondence will be possible since they provide interpolating functions between weak and strong coupling regimes. Examples of this type of observables are provided by the chiral primary operator or by the supersymmetric (BPS) Wilson loops that will be the central theme of this thesis.

The relation of $\mathcal{N} = 4$ SYM with the string theory is the most but not certain the only important feature of the theory. Another nice property is that it is supposed to be invariant under a generalized electric-magnetic duality called S-duality. Historically this conjecture has been originated by the work of Montonen and Olive [17] who proposed that in some gauge theory there is an electric-magnetic symmetry that inverts the gauge coupling and replaces the group G with the so called Langlands dual group ${}^L G$ in which the root and the weight lattices are exchanged. The simple situation in which this duality is realized, is the $\mathcal{N} = 4$ SYM theory

where elementary electrons and monopoles belong to the same multiplet and have the same quantum numbers. In the eighties it was discovered that this electric-magnetic duality can be extended to a more general $SL(2, \mathbb{Z})$ symmetry acting on the generalized coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad (1.3)$$

as

$$\tau \rightarrow \tilde{\tau} = \frac{a\tau + b}{c\tau + d} \quad (1.4)$$

with $ad - bc = 1$. The full duality states

$$\begin{aligned} \mathcal{N} = 4 \text{ SYM with gauge group } G \text{ and coupling constant } \tau \\ \equiv \\ \mathcal{N} = 4 \text{ SYM with gauge group } {}^L G \text{ and coupling constant } \tilde{\tau} \end{aligned}$$

This correspondence defines an isomorphism between operators in the two theories. As example, since it exchanges the electric with the magnetic degrees of freedom, it maps the Wilson loop that describes an electric charge moving on a closed path C with 't Hooft loop that instead describes a monopole passing through the same contour. Several progresses have been done in the understanding the action of S-duality especially on chiral primary operators [18, 19, 20], on surface operators [21, 22, 23] and on Wilson/'t Hooft loops [24, 25, 26]. In the perspective of the Maldacena correspondence, the S-duality has a natural manifestation on the string side. Indeed type IIB superstring theory on a curved background is conjectured to have an identical $SL(2, \mathbb{Z})$ self-duality symmetry acting on the generalized string coupling constant τ as usual

$$\tau = \frac{\langle C \rangle}{2\pi} + \frac{i}{g_s} \rightarrow \tilde{\tau} = \frac{a\tau + b}{c\tau + d} \quad (1.5)$$

and that exchanges fundamental strings with D1-strings, D5-branes with NS5-branes and take the D3-branes into themselves.

Actually we can tell more about S-duality in $\mathcal{N} = 4$ SYM. In a very interesting series of paper, E. Witten and collaborators [21, 27] show that it is related to the so called Langlands duality that is a kind of unified scheme for many results in number theory and specifically to the geometric version of this duality that involves curves over ordinary complex Riemann surfaces (the so called geometric Langlands program). The connection between these dualities is based on a dimensional reduction of a twisted version of $\mathcal{N} = 4$ SYM to a family of topological sigma-models in two dimensions on a Riemann surface Σ [28]. This $\mathcal{N} = (4, 4)$ two-dimensional sigma-model obtained has a T-duality that corresponds to the $\tau \rightarrow 1/\tau$ transformation in

the gauge theory. Since the geometric Langlands correspondence is based on the action of this T-duality on some eigenbrane, in a sense it arises from the electric-magnetic duality. For more details on this subject we refer the reader to [27] and reference therein.

The story is not over yet and the $\mathcal{N} = 4$ SYM reserves us other surprises. Other elegant structures have been observed in this theory; we refer to the anomalous dimensions of the composite operators and to the n -particles scattering amplitudes. An impressive development for what concerns the anomalous dimensions in $\mathcal{N} = 4$ SYM have been done with the discovery of the integrability in the planar limit (large N limit). In their seminal paper [29] Minahan and Zarembo showed that the planar dilation operator of $SO(6)$ scalar sector of $\mathcal{N} = 4$ SYM at one loop can be identified with the Hamiltonian of an integrable $SO(6)$ spin-chain. So the problem of computing spectrum of local operator (the diagonalization of the dilation operator) can be reformulated as the diagonalising of an integrable spin chain Hamiltonian. Afterwards using Bethe-Ansatz methods, the work of Minahan and Zarembo has been generalized to the full $PSU(2,2|4)$ sector by Beisert and Staudacher [97].

One of the most spectacular result in the integrability framework was made by Beisert, Eden and Staudacher studying the asymptotic Bethe-Ansatz for the anomalous dimensions of composite operators in a $SL(2)$ sector of the $\mathcal{N} = 4$ SYM. They have arrived to an astonishing result concerning the so-called cusp anomaly, whose nonperturbative expression was found to be encoded into an exact integral equation (BES equation) [30]: the weak coupling perturbative solution agrees with the Feynman diagrams expansion [31] while the strong coupling asymptotic solution [32, 33, 34, 35, 36] reproduces the sigma-model result obtained from string theory [37, 38]. Recently another very intriguing development has been done with the proposal of the so called Y-system. Indeed in [39, 40] the authors propose a set of equations which determine the anomalous dimensions of composite operators for any length of the operator and at any coupling constant g_{YM} in the large N limit of the $\mathcal{N} = 4$ SYM.

The AdS/CFT correspondence leads to expect integrability on the string side as well where the scaling dimensions of the gauge theory are identified with the energies of the dual strings. The classical string sigma model on $AdS_5 \times S^5$ is integrable and this property becomes manifest with the existence of the Lax Pairs. For a comprehensive review on the subjects see for example [86] and reference therein.

Another domain where new and unexpected structures have emerged is that of scattering amplitudes. For the so-called maximally helicity violating (MHV) gluon scattering amplitudes, namely those in which only two gluons have a certain helicity while the others have the opposite one, a simple expression holds. Indeed all tree level MHV amplitudes satisfy particular recursion relations derived by Britto, Cachazo, Feng and Witten [42, 43]. These relations can be solved to yield a closed form for the MHV amplitudes at tree-level. At higher loops, MHV gluon amplitudes

have been calculated using unitary-based techniques and a direct evaluation of the integrals exhibits a surprising iterative structure [45]. In fact Bern, Dixon and Smirnov (BDS) suggested that the rescaled n -point MHV amplitude is given by

$$\mathcal{M}_n = \exp \left[\sum_{l=1}^{\infty} a^l (f_0^l(\epsilon) + \epsilon f_1^l(\epsilon) + \epsilon^2 f_2^l(\epsilon)) M_n^1(\epsilon) + C^l + \mathcal{O}(\epsilon) \right] \quad (1.6)$$

where the coefficients f_0^l are the coefficients of the cusp anomalous dimension

$$f(\lambda) = \sum_l a^l(\lambda) f_0^l, \quad (1.7)$$

f_1 and f_2 are in relations with the collinear anomalous dimension and M_n^1 is the n -point function at one loop.

This surprising relation has been successively shown to hold only for the four and five points amplitudes, while for the case of six or more external particles this relation has to be modified adding a function of the conformal ratios [46, 47]. Motivated by the BDS ansatz, the strong coupling analysis of gluon amplitude using string theory tools has been performed in two interesting work of Alday and Maldacena (AM) [48, 49]. Here the authors suggested that the planar gluon amplitudes in this regime can be calculated solving the problem of a minimal surface that stretches in the AdS space and that is bounded by the polygon C_n made of n light-like segments $[x_i, x_{i+1}]$, with the coordinates x_i related to the on-shell gluon momenta by the relation $x_i^\mu - x_{i+1}^\mu = p^\mu$. Summarizing they proposed

$$\log(M_n) = -\frac{\sqrt{\lambda}}{2\pi} A_{min}(C_n). \quad (1.8)$$

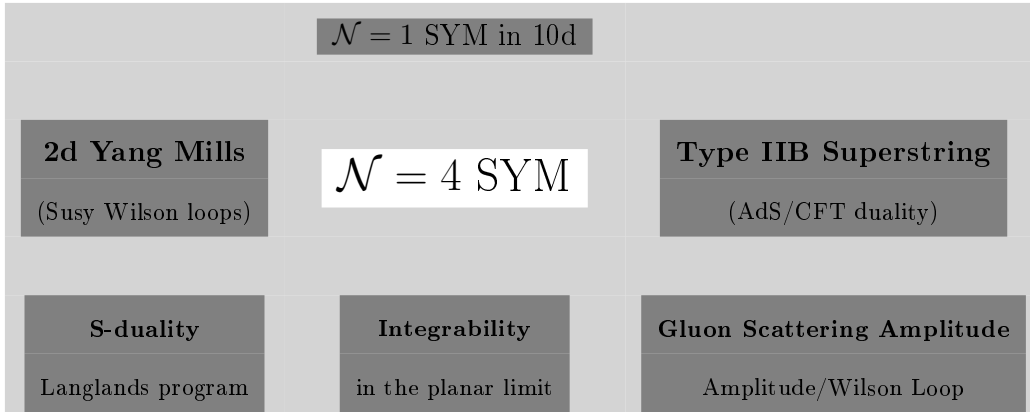
The above relation makes clear that the strong coupling computation of the scattering amplitude coincide with the calculus of the expectation value of a Wilson loop $W(C)$ defined on the light-like contour C made by the n -segments $[x_i, x_{i+1}]$.

Afterwards the gluon amplitude/Wilson loop duality proposed by AM has been conjectured to hold also at weak coupling [50, 51]. Explicitly we have

$$\log \left(1 + \sum_{l=1}^{\infty} a^l M_n^l \right) = \log \left(1 + \sum_{l=1}^{\infty} a^l W_n^l + \mathcal{O}(\epsilon) \right) \quad (1.9)$$

where M_n^l and W_n^l are respectively the l -loop correction to the MHV amplitude and to the VEV of a Wilson loop defined on C . The above equality also implies that the gluon scattering amplitudes inherit the space-time conformal invariance possessed by the Wilson-loop. This new symmetry of the amplitudes is not the original conformal invariance of the theory, because it acts in momentum space

and it is realized by local operators in [50]. It is usually called *dual conformal symmetry* to distinguish it from the other. Quite surprisingly when one commutes the conformal with the dual conformal generators a particular algebraic structure emerges [52]. The full $PSU(2, 2|4)$ symmetry of the theory is in fact lifted to a Yangian (the same structure appears also in the study of the spectrum of anomalous dimensions). For a comprehensive review on the gluon scattering amplitudes and their relation with Wilson loops see [53] and reference therein.



$\mathcal{N} = 4$ SYM in summary

1.2 The Wilson loops in $\mathcal{N} = 4$ SYM

This thesis is devoted to study supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM. As observed in the previous section these observables are interesting essentially for two reasons: the first one is that in some cases they can be useful tools in checking the AdS/CFT duality and the second one is that they are related to the gluon scattering amplitudes. In this work we discuss these observables and their general properties with particular attention to the first point. Indeed we will present different exactly calculable operators; we will be able to derive a general result for their *VEV* valid at any value of N and at any coupling constant g and to check the results both at perturbative and non-perturbative level.

A notable example of such operators is the half BPS circular Wilson loop [64, 65, 98]. In that case it has been rigorously proven with a localization procedure [66] that its *VEV* is fully captured by a hermitian matrix model with a gaussian potential. On the gauge side the Feynman diagrams computation performed in [64] coincides with the perturbative expansion of the matrix model. On the string side the dual operators are minimal surface in AdS_5 ending on the loops at the holographic boundary and their expectation value can be obtained by calculating

the string action after a suitable regularization. Quite surprisingly the result match exactly with the large λ limit of the matrix model.

It is important to extend this example to a richer class of observables that preserve a less amount of supersymmetry but still exactly solvable. Interesting operators with this properties were considered in [58, 59, 60]. These Wilson loops, defined for an arbitrary contour lying on S^3 , couple to only three of the six scalars of the $\mathcal{N}=4$ SYM and generically preserve only two supercharges (a linear combination of the Poincaré and superconformal ones).

The situation becomes more interesting when we restrict the loops on a two sphere inside S^3 . Indeed these operators (1/8 BPS) exhibit very nice features : their expectation value is supposed to be captured by the analogous computation in the zero instanton sector of the ordinary bosonic YM_2 on a two sphere. The two dimensional Yang-Mills theory is an almost topological theory, namely it is invariant under area preserving diffeomorphism, and is exactly solvable. Here the VEV of the Wilson loops can be computed and the solution is given by the gaussian matrix model with a rescaling of the coupling constant by a factor that depends from the areas delimited by the loops.

BPS Wilson loops on S^2 in $\mathcal{N}=4$ SYM \equiv Wilson loops on S^2 in the zero instanton sector of YM_2

In our work we have found some interesting checks of this conjecture. The first one is the two-loop non-trivial perturbative computation illustrated in chapter 3 [75]. After a carefully analysis about the cancelation of divergences that appear in the intermediate steps of computation, we have sum the ladder with the interaction diagrams for a particular circuits on S^2 (a "wedge") finding a good numerical agreement with the matrix model prediction. Differently from the half BPS circular case, here the interactions do not cancel (at least in Feynman gauge) but have an intriguing interplay with the ladder diagrams to reproduce the matrix model result. Other checks have been proposed in literature; we briefly review them in the rest of the chapter 3 (a strong coupling computation and the derivation of the matrix model from localization procedure).

Another/but similar class of Wilson operators exactly solvable are presented in chapter 4. They lie on a hyperbolic three space \mathbb{H}_3 and they are related to the previous one by a Wick rotation of the scalar couplings. Generically they are 1/16 BPS but when they lie on a \mathbb{H}_2 inside \mathbb{H}_3 , the restriction enhances the preserved supercharges to four. Also here a gaussian matrix model is supposed to capture the VEV of these operators at least if the contour is finite (on \mathbb{H}_2 there are non-

compact loop but in that case we don't know if and how the reduction holds). The chapter four will be dedicated to give a perturbative two-loop evidence of the proposal [113].

In [78, 79] we have extended the relation $\mathcal{N} = 4$ SYM/ YM_2 to the case of the connected correlator of two Wilson loops.

The Correlator of two Wilson loops on S^2 in $\mathcal{N} = 4$ SYM
 \equiv
The Correlator of two Wilson loops on S^2
in the zero instanton sector of YM_2

As a first step we have computed the two matrix model governing the correlator in the zero instanton sector of YM_2 . Then we have tried to demonstrate the equivalence by means of some perturbative and non-perturbative checks. We have performed an hard perturbative computation (up to g^6 order) finding a good numerical agreement for two explicit configurations. When one of the two circuits shrinks to zero the numerical computation becomes more and more slowly and we have considered another approach to check the conjecture. In this limit the correlator can be computed by considering the operator product expansion (OPE) of the Wilson loop. Each operator in the expansion contributes to the correlator with a prefactor that goes like the shrinking radius to its conformal dimension. Therefore the expansion at small coupling generically produces logarithms in the shrinking radius if quantum corrections modify the classical dimension. Since these logarithmic terms cannot be reproduced by the matrix model proposal, their presence will then indicates a failure of our conjecture. Fortunately this is not the case. Indeed with a very non-trivial cancelation all log terms cancel in the g^6 computation and thus only non "dangerous" contributions survive in this limit (as predicted).

Finally we have performed a strong coupling computation. In this regime, taking the limit in which the separation between the Wilson loops is much larger than their size, namely the situation in which the two circles migrate to the poles of the sphere, one has to calculate the exchange of the light supergravity modes between the two worldsheets that describe the Wilson loops in this dual picture. We have found only a partial matching with the matrix model. Indeed there should be a non-trivial cancelation between the light and the more heavier supergravity modes that contribute in the computation but we are not able to reproduce correctly this vanishing.

Below, there is the list of my original works discussed in this thesis :

1. "*Supersymmetric Wilson loops at two loops*",
A. Bassetto, L. Griguolo, F. Pucci, D. Seminara, JHEP **0806** (2008) 083,
arXiv:hep-th/0804.3973
2. "*Correlators of Supersymmetric Wilson-loops, protected operators
and matrix model in $\mathcal{N}=4$ SYM*"
A. Bassetto, L. Griguolo, F. Pucci, D. Seminara, S. Thambyahpillai and
D. Young, JHEP **0908** (2009) 061, arXiv:hep-th/0905.1943
3. "*Correlators of Supersymmetric Wilson-loops in $\mathcal{N}=4$ SYM at
weak and strong coupling*"
A. Bassetto, L. Griguolo, F. Pucci, D. Seminara, S. Thambyahpillai and
D. Young, JHEP **1003** (2010) 038, arXiv:hep-th/0912.5440
4. "*BPS Wilson Loops on Hyperbolic space*,"
F. Pucci, *to appear*

Chapter 2

BPS Wilson Loops in $\mathcal{N} = 4$ SYM

2.1 Wilson loops: from QCD to $\mathcal{N} = 4$ SYM

Wilson loops are interesting non-local objects usually defined in any gauge theory. They have been proposed by K.Wilson as an order parameter in the lattice formulation of quantum chromodynamics [54]. Essentially they measure the response of the gauge field to an external quark-like source passing around a closed contour C . In a more mathematical language they are the path-ordered exponential of the connection A_μ along C ¹:

$$\langle W(C) \rangle = \frac{1}{\dim(R)} \text{Tr}_R \left[\mathcal{P} \exp \left(i \oint_C A_\mu dx^\mu \right) \right]. \quad (2.1)$$

In QCD these operators in the fundamental representation F are used to distinguish the different phases of the theory. Indeed from the expectation value of a particular Wilson loop one can extract the quark-antiquark potential. Taking a rectangular contour $C = L \times T$, where L and T are respectively the space and the time extension of the loop, it is easy to see that $\langle W(C) \rangle$, in the large T limit, behaves as

$$\langle W(C) \rangle_{T \rightarrow \infty} \sim e^{-TV(L)}. \quad (2.2)$$

Here $V(L)$ is the gauge field energy associated to the static quark-antiquark source. The quarks are then confined if this potential, in the large separation limit, goes as

$$V(L) \sim \sigma L \quad (2.3)$$

¹ Above R denotes the representation of the gauge group, where the quark-like external source transforms.

where σ is the (QCD) string tension. Lattice simulations show for its behavior a linear rise at large L and a Coulomb behavior at small L (as expected).

To be precise these considerations hold only for pure gauge theories. For full QCD the situation is more involved since the linear rising of the potential is screened. The flux string between the quarks breaks when the energy is large enough to pop an additional quark-antiquark couple out of the vacuum and to create two mesons. This phenomenon of the screening can be tested by plotting $V(L)$ for different values of the dynamical quark mass and observing the flattening of the linear rise.

In the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with gauge group $U(N+1)$ the situation is quite different since no field transforms in the fundamental representation of the gauge group, but they all belong to the adjoint. In other words, there is no simple analog of the massive quarks.

In order to introduce a massive source in the fundamental representation, one has to break the original $U(N+1)$ gauge symmetry to $U(N) \times U(1)$ giving a non-trivial expectation value ϕ_0 to one of the scalar fields. Then the spectrum will contain W -bosons with a mass proportional to $|\phi_0|$ and transforming in the fundamental representation of $U(N)$. So in the limit $|\phi_0| \rightarrow \infty$ they provide the very massive “sources” necessary to define a sensible Wilson loop in the $U(N)$ theory.

The Wilson loop in $\mathcal{N} = 4$ SYM can be then defined as the phase factor acquired by an infinitely massive W -boson moving along a closed path C . Quite naturally the operator that we obtain [55, 56] couples not only with the gauge field but also with the six scalars of the theory and it has the form

$$\langle W(C) \rangle = \frac{1}{N} \text{Tr} \left[\mathcal{P} \exp \left(\oint_C (iA_\mu \dot{x}^\mu + \phi^I \theta^I(s) |\dot{x}|) ds \right) \right], \quad (2.4)$$

where the couplings θ^I ($I = 1 \dots 6$) with $\theta^I \theta^I = 1$ map each point of the loop to a point on a five-sphere. The additional scalar couplings in (2.4) become natural if we think to the $\mathcal{N} = 4$ SYM as the dimensional reduction of the $\mathcal{N} = 1$ SYM in ten dimensions. Indeed if we consider a Wilson loop of the form (2.1) in ten dimensions, the dimensional reduction naturally yields

$$\langle W(C) \rangle = \frac{1}{N} \text{Tr} \left[\mathcal{P} \exp \left(\oint_C (iA_\mu \dot{x}^\mu + \phi^I y^I(s)) ds \right) \right], \quad (2.5)$$

where we have used that the original gauge field A_M breaks into a gauge vector A_μ and six scalars ϕ^I and we have denoted with y^I the coordinates in the six transverse dimensions. In (2.5) the path $y^I(s)$ in the transverse directions do not need to be closed to ensure gauge invariance, since the gauge transformations depend only on the space-time coordinates x^μ . To recover the form (2.4), it is sufficient to set $y^I(s) = |\dot{x}| \theta^I(s)$.

The meaning of the condition $\theta^I \theta^I = 1$ can be understood if we consider the behavior of the Wilson loop (2.4) under the supersymmetry transformations:

$$\delta_\epsilon A_\mu = \bar{\Psi} \Gamma_\mu \epsilon \quad \text{and} \quad \delta_\epsilon \phi_I = \bar{\Psi} \Gamma_I \epsilon. \quad (2.6)$$

In order to avoid cumbersome expressions, we have adopted a ten dimensional notation. The four Weyl fermions ψ_I of $\mathcal{N} = 4$ SYM and the supersymmetry parameters ϵ_I have been collected into two ten-dimensional Majorana-Weyl spinors Ψ and ϵ . Moreover, we have denoted with $\Gamma_M = (\Gamma_\mu, \Gamma_I)$ the usual Dirac matrices in ten dimensions. Then the supersymmetry variation of the Wilson loop (2.4) reads

$$\begin{aligned} \delta_\epsilon W(C) = \frac{1}{N} \text{Tr} \left[\mathcal{P} \int_C ds \bar{\Psi} (i\Gamma_\mu \dot{x}_\mu + \Gamma^I \theta^I |\dot{x}|) \epsilon \times \right. \\ \left. \times \exp \left(\oint_C (iA_\mu \dot{x}^\mu + \phi^I \theta^I(s) |\dot{x}|) ds \right) \right]. \end{aligned} \quad (2.7)$$

Some part of the supersymmetry will be preserved if the condition

$$(i\Gamma_\mu \dot{x}_\mu + \Gamma^I \theta^I(s) |\dot{x}|) \epsilon = 0 \quad (2.8)$$

admits a non trivial solution for ϵ . Consistency requires that

$$(i\Gamma_\mu \dot{x}_\mu + \Gamma^I \theta^I(s) |\dot{x}|)^2 \epsilon = 0 \quad \Rightarrow \quad |\dot{x}|^2 (1 - \theta^I \theta^I) = 0. \quad (2.9)$$

Therefore the condition $\theta^I \theta^I = 1$ is necessary to endow these Wilson loops with a certain amount of supersymmetry. This constrain will also ensure a regular behavior of the effective propagator at coincident points, namely

$$\begin{aligned} \langle (iA_\mu \dot{x}_1^\mu + \phi^I \theta^I(s_1) |\dot{x}_1|) (iA_\mu \dot{x}_2^\mu + \phi^I \theta^I(s_2) |\dot{x}_2|) \rangle = \\ = -\frac{1}{2} \left[\theta^I(s_1) \ddot{\theta}^I(s_1) - \frac{(\dot{x}_1 \cdot \ddot{x}_1)^2}{(\dot{x}_1 \cdot \dot{x}_1)^2} + \frac{(\ddot{x}_1 \cdot \dot{x}_1)}{(\dot{x}_1 \cdot \dot{x}_1)} \right] + O((s_1 - s_2)). \end{aligned} \quad (2.10)$$

In other words, for regular contours, which do not contain cusps, the typical power-like singularity of the propagator for $s_2 \rightarrow s_1$ has been smoothed out.

However the constraint (2.9) is not sufficient to ensure an actual invariance of the loop under supersymmetry transformations. Since the spinor ϵ solving (2.9) may depend arbitrarily on the parameter s of the contour C , it does not generate, in general, a super-conformal transformation. In this case, with an abuse of language, we shall say that the operator is only *locally supersymmetric*.

To have a BPS (*i.e.* supersymmetric) Wilson loop, the dependence on s must be of the form ²

$$\epsilon(s) = \epsilon_0 + x^\mu(s) \Gamma_\mu \epsilon_1, \quad (2.11)$$

where ϵ_i are constant spinors. This requirement yields constraints either on the loop (\dot{x}_μ) or on the scalar couplings ($\theta^I(s)$) or on both quantities.

²Let us recall that a superconformal transformation on \mathbb{R}^4 is generated by a spinor of the form $\epsilon = \epsilon_0 + x^\mu \Gamma_\mu \epsilon_1$. The constant spinor ϵ_0 is responsible for the Poincaré part, while the constant spinor ϵ_1 for the conformal part.

2.2 BPS Wilson loops in $\mathcal{N} = 4$ SYM : Supersymmetric properties

A simple idea to solve eq. (2.8) and to obtain a supersymmetric Wilson loop for an arbitrary shape of the contour was proposed by Zarembo in [57]. The construction is based on the additional requirement that the position of the loop on the five sphere defined by the vector $\theta^I(s)$ follows the tangent vector of the contour C . Namely one chooses

$$\theta^I(s) = M_\mu^I \frac{\dot{x}^\mu}{|\dot{x}|} \quad (2.12)$$

with M_μ^I a rectangular 4×6 matrix that satisfies $M_\mu^I M_\nu^I = \delta_{\mu\nu}$ (we do not need to give its explicit form since $SO(4) \times SO(6)$ is a global symmetry of the theory). Substituting this ansatz into the equation (2.8) and considering only super-Poincaré transformations all s-dependence factors out

$$\dot{x}^\mu (\Gamma_\mu - i M_\mu^I \Gamma^I) \epsilon = 0. \quad (2.13)$$

and for a contour whose tangent vector spans \mathbb{R}^4 we remain with four independent algebraic equations

$$(\Gamma_\mu - i M_\mu^I \Gamma^I) \epsilon = 0. \quad (2.14)$$

These four equations are consistent and the subspace of the solutions has dimensions 1. We obtain a Wilson loop which preserves 1/16 of the original Poincaré supersymmetry.

If the curve lies in a lower dimensional subspace of \mathbb{R}^4 , the requirement (2.14) is too strong: we have an enhancement of the supersymmetry according to the dimension of the subspace. The line is 1/2 BPS; if the path belongs to \mathbb{R}^2 , the loop is 1/4 BPS; if it is contained in \mathbb{R}^3 , we obtain 1/8 BPS operator.

Another interesting proposal to solve (2.8) has been put forward by Drukker, Giombi, Ricci and Trancanelli (DGRT) in [58, 59, 60]. There, the authors consider an interesting new class of Wilson loops of arbitrary shape and defined on a three sphere S^3 . We shall not discuss them here in detail since a complete analysis of this class of operators and of its supersymmetric properties will be presented in chap. 3.

The above two families do not exhaust all the possible supersymmetric Wilson loops with only bosonic couplings. Recently a systematic classification of all the solutions of (2.8) has been obtained by Pestun and Dymarsky in [61]. Their first step was to recast the eq. (2.8) in a full ten dimensional language by introducing the vector $v^M = \{\frac{dx^\mu}{ds}, \theta^I(s)\}$. One finds

$$v^M(x) \Gamma_M \epsilon(x) = 0 \quad (2.15)$$

where, as usual, $\Gamma_M = (\Gamma_\mu, \Gamma_I)$ are the ten dimensional gamma matrices. Then to solve this system of equations, they fix ϵ and look for the couplings v^M so that eq.

(2.15) is satisfied. We can distinguish two different types of solutions depending on whether or not the vector $u^M = \epsilon \Gamma^M \epsilon$ vanishes.

When $u^M(\epsilon) \neq 0$ there is a unique solution of eq. (2.15) and it is given by $v^M = \lambda u^M$ with λ a complex number [61]. The resulting circuits are the orbits of the conformal transformations generated by $Q_{\epsilon(x)}^2$. If we consider only closed loops, we obtain operators defined on (p, q) *Lissajou figures* where p and q are integer numbers.

If u^M is identically zero, the spinor ϵ is not generic, but it is a *pure spinor* [62]. This kind of spinor is annihilated by the largest possible subspace of the ten dimensional Clifford Algebra, namely by “half” of the gamma-matrices Γ^M . This property can be used to endow $\mathbb{R}^{10} \times \mathbb{C}$ with an almost complex structure J_ϵ [63]. The solutions of (2.15) are then antiholomorphic vectors with respect to J_ϵ . More generically the authors show that the interesting solutions can be found when u^M is pure in a subspace Σ_ϵ of \mathbb{R}^4 . In fact in that case one can always find a set of scalar couplings that makes supersymmetric the Wilson loop supported by any curve in Σ_ϵ .

With the above method the authors of [61] reduce the classification of all the possible supersymmetric Wilson operators to the study of the subspace Σ_ϵ of \mathbb{R}^{10} which admit a pure spinor. The well-known DGRT and Zarembo’s Wilson loops can be recovered as a particular limit of these new interesting operators.

2.3 The circular Wilson loops

Supersymmetric Wilson loops can be sometimes computed in a closed form, namely to all orders in the ’t Hooft coupling. For example, it is relatively easy to show that the vacuum expectation value (*VEV*) of all the Zarembo’s loops is trivial and it is given by 1, independently of the shape of the contour³.

The first and best-known example of BPS Wilson loop which possess a non-trivial *VEV* is the circular one discussed in [64, 65, 66]. Its value can be exactly computed providing us a quantity which smoothly interpolates between the weak and strong coupling regime. The result for large ’t Hooft coupling can be thus compared with the prediction of the AdS/CFT correspondence.

In the following we shall review some features and properties of this observable, starting from its construction and proceeding then to its explicit evaluation, both from gauge and string theory side. This operator is a Maldacena-Wilson loop (2.4) defined on a circuit parameterized by

$$x_\mu(\tau) = (\cos \tau, \sin \tau, 0, 0) \quad (2.16)$$

³From the point of view of the correspondence this result is not immaterial, in fact it states that there is an entire family of minimal area problems for which $A_{min} = 0$. This was explicitly verified in [69].

with $0 < \tau < 2\pi$ and with the scalar couplings $\theta^I(s)$ constants θ_0^I that essentially describe the position of the loop on the five sphere. Using the $SO(6)$ R -symmetry it is immediately to see that effectively the circuit couples to only one of the six scalars of the theory, say $\phi_0 \equiv \theta_0^I \phi_I$. Thus the operator can be written as

$$\langle W(C) \rangle = \frac{1}{N} \text{Tr} \left[\mathcal{P} \exp \left(\oint_c (iA_\mu \dot{x}^\mu + \phi_0) ds \right) \right]. \quad (2.17)$$

The equation (2.8) for the super-conformal spinors,

$$\epsilon(x) = \epsilon_0 + x_\mu \Gamma^\mu \epsilon_1, \quad (2.18)$$

which preserve the loop (2.17), is solved by choosing

$$\epsilon_1 = -i\theta_0^I \Gamma_{12} \Gamma_I \epsilon_0. \quad (2.19)$$

The loop is not invariant under Poincaré ($\epsilon_1 = 0$) or conformal ($\epsilon_0 = 0$) transformations separately, but only under a combination of them. This is different from what occurs in the case of Zarembo's operators. Since (2.19) simply expresses ϵ_1 in terms of the ϵ_0 which is still arbitrary, we are dealing with a loop, which is 1/2 BPS.

Let us try to calculate its VEV perturbatively. The first ingredient to compute is the effective propagator $\langle (iA_\mu(x_1)\dot{x}_1^\mu + \phi_0(x_1))(iA_\mu(x_2)\dot{x}_2^\mu + \phi_0(x_2)) \rangle$:

$$\begin{aligned} \langle (iA_\mu(x_1)\dot{x}_1^\mu + \phi_0(x_1))(iA_\mu(x_2)\dot{x}_2^\mu + \phi_0(x_2)) \rangle &= \frac{1}{4\pi^2} \frac{1 - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{(x(\tau_1) - x(\tau_2))^2} = \\ &= \frac{1}{4\pi^2} \frac{1 - \cos(\tau_1 - \tau_2)}{2(1 - \cos(\tau_1 - \tau_2))} = \frac{1}{8\pi^2}, \end{aligned} \quad (2.20)$$

Since this propagator is constant, the contribution of all the ladder diagrams, namely of all the graphs which do not contain any internal vertex, can be summed with the help of the following gaussian matrix model

$$W_{\text{ladder}} = \frac{1}{Z} \int \mathcal{D}M \frac{1}{N} \text{Tr} \exp(M) \exp \left(-\frac{2}{g^2} \text{Tr} M^2 \right), \quad (2.21)$$

where

$$Z = \int \mathcal{D}M \frac{1}{N} \text{Tr} \exp \left(-\frac{2}{g^2} \text{Tr} M^2 \right). \quad (2.22)$$

and M is an $N \times N$ Hermitian matrix.

Next we have to take into account the diagrams with internal vertices. To begin with, we shall consider the order g^4 . We have two families of contributions: [a] *spider diagrams* and [b] the *bubble diagrams*, all depicted in fig. (2.1). It is possible to show that the sum of all these graphs is not only finite but it vanishes as observed firstly in [64].

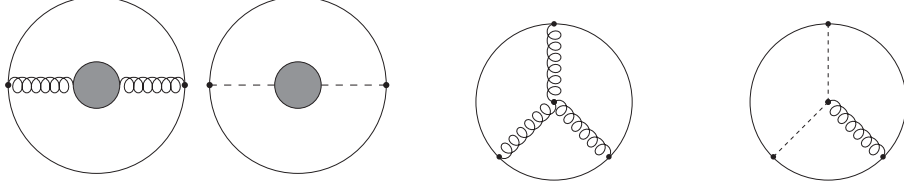


Figure 2.1: One-loop correction to the gluon and the scalar exchange and spider-diagrams, gauge and scalar contributions.

In [64, 65] it was conjectured that the above cancellation for the diagrams containing internal vertices extends to all order in g^2 . In other words, the value of the circular Wilson loop is only captured by the ladder diagrams and it is simply given by the matrix model (2.21).

The actual value of the matrix integral can be computed quite easily by means of the orthogonal polynomial technique, as illustrated in [65]. One finds

$$\langle W_{\text{circle}} \rangle = \frac{1}{N} \sum_{j=0}^{N-1} L_j \left(-\frac{\lambda}{4N} \right) \exp \left[\frac{\lambda}{8N} \right] = \frac{1}{N} L_{N-1}^1 \left(-\frac{\lambda}{4N} \right) \exp \left[\frac{\lambda}{8N} \right] \quad (2.23)$$

where L_n^m are the Laguerre Polynomials

$$L_n^m(x) = \frac{1}{n!} \exp(x) x^{-m} \frac{d}{dx^n} (\exp(-x) x^{n+m}) \quad (2.24)$$

and $\lambda = g^2 N$ is the 't Hooft couplings.

Let us consider the large N -limit of this expression, where only planar diagrams survive. Recalling that

$$\lim_{n \rightarrow \infty} \left[n^{-\alpha} L_n^\alpha \left(\frac{x}{n} \right) \right] = x^{-\alpha/2} J_\alpha(2\sqrt{x}), \quad (2.25)$$

we find the complete expression of our Wilson loop at large N :

$$\langle W_{\text{circle}} \rangle_{\text{Large } N} = \frac{2}{\sqrt{-\lambda}} J_1(\sqrt{-\lambda}) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}). \quad (2.26)$$

We can easily take the strong-coupling limit of this expression and for the leading contribution we find

$$\langle W_{\text{circle}} \rangle_{\text{Large } N} \simeq_{\lambda \rightarrow \infty} \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}}. \quad (2.27)$$

It would be interesting to compare this prediction with the same computation performed on the string side. This issue will be discussed in sec. 2.5.

2.4 Localization for the circular Wilson loop

The conjectured value (2.23) for the circular Wilson loop can be proven to be correct by exploiting the so-called *localization procedure* [66]. This method is a powerful and general tool in supersymmetric gauge theories. When it can be successfully applied, the quantum average of the observable receives contributions only from a particular subset of the classical configurations.

The localization can be explained briefly as follows. Let's consider a theory defined by an action S that is invariant under a fermionic symmetry Q

$$QS = 0. \quad (2.28)$$

Let $Q^2 = \mathcal{L}$ be some compact bosonic symmetry of the theory and \mathcal{T} a Q -closed operator ($Q\mathcal{T} = 0$). If we deform the action with a Q -exact term

$$S \rightarrow S + tQV. \quad (2.29)$$

the expectation value of \mathcal{T} doesn't change. Indeed

$$\frac{d}{dt}\langle\mathcal{T}\rangle_t = \langle\mathcal{T}\{Q, V\}\rangle = \langle\{Q, \mathcal{T}V\}\rangle = 0. \quad (2.30)$$

and so

$$\langle\mathcal{T}\rangle_t = \langle\mathcal{T}\rangle, \quad (2.31)$$

the expectation value of \mathcal{T} in the deformed theory is the same as in the undeformed one. If now t goes to infinity and the term added tQV is semipositive defined, we observe that the theory has to *localize* (by semiclassical arguments) on some set of critical points of QV over which we have to sum. Since we can take t strictly equal to infinity, the contribution of each critical point to the path-integral is simply given by the one-loop saddle point approximation.

Coming back to the $\mathcal{N} = 4$ SYM, we could use the localization to calculate the VEV of the 1/2 BPS Circular Wilson loop. To begin with we shall consider the euclidean version of the theory on a four sphere (with radius r) obtained via dimensional reduction of the euclidean $\mathcal{N} = 1$ SYM 10d and described, using the ten dimensional notation, by the action

$$S_{\mathcal{N}=4} = \frac{1}{2g_{YM}^2} \int_{S^4} \sqrt{g} d^4x \left(\frac{1}{2} F_{MN} F^{MN} - \bar{\Psi} \Gamma^M D_M \Psi + \frac{2}{r^2} \phi^A \phi_A \right) \quad (2.32)$$

where the last term in eq. (2.32), added to the usual action in flat-space, is necessary to preserve conformal invariance. The theory is invariant under the superconformal transformations

$$\delta_\epsilon A_M = \bar{\Psi} \Gamma_M \epsilon \quad (2.33)$$

$$\delta_\epsilon \Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + \frac{1}{2} \Gamma_{\mu A} \phi^A \nabla^\mu \epsilon \quad (2.34)$$

where ∇^μ is the covariant spinorial derivative and

$$\epsilon = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\epsilon_0 + x^\mu \Gamma_\mu \epsilon_1) \quad (2.35)$$

is the conformal killing spinor on S^4 in stereographic coordinates. At this step the superconformal algebra closes only on-shell [66] but to use the localization method we need an off-shell closure of the fermionic subalgebra (at least for the charge that we use in the localization procedure). To close off-shell the relevant supersymmetry of the $\mathcal{N} = 4$ theory on S^4 we make the dimensional reduction of Berkovits method [68] used for the ten-dimensional $\mathcal{N} = 1$ SYM. The number of auxiliary fields compensates the difference between the number of fermionic and bosonic off-shell degrees of freedom modulo gauge transformations. In the $\mathcal{N} = 4$ case we add $16 - (10 - 1) = 7$ auxiliary fields K_i with free quadratic action and modify the super-conformal transformations to

$$\delta_\epsilon A_M = \bar{\Psi} \Gamma_M \epsilon \quad (2.36)$$

$$\delta_\epsilon \Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + \frac{1}{2} \Gamma_{\mu A} \phi^A \nabla^\mu \epsilon + K_i \nu^i \quad (2.37)$$

$$\delta_\epsilon K_i = -\Psi \Gamma^M D_M \nu_i \quad (2.38)$$

with the seven ten-dimensional Majorana-Weyl (commuting) spinors ν^i ($i=1\dots 7$) that satisfy

$$\epsilon \Gamma_M \nu_i = 0, \quad \frac{1}{2} (\epsilon \Gamma_N \epsilon) \Gamma_{\alpha\beta}^N = \nu_\alpha^i \nu_\beta^i + \epsilon_\alpha \epsilon_\beta, \quad \nu_i \Gamma^M \nu_j = \delta_{ij} \epsilon \Gamma^M \epsilon. \quad (2.39)$$

At this point we can evaluate the circular Wilson loop, following the general picture briefly illustrated at the beginning of this section. First, we choose one of the super-conformal spinor $\epsilon(x)$ defined by (2.18) and (2.19) by selecting an ϵ_0 , which satisfies

$$\Gamma^{1234} \epsilon_0 = -\Gamma^{5678} \epsilon_0 = -1 \quad \bar{\epsilon}_0 \epsilon_0 = 1. \quad (2.40)$$

Next we shall denote the super-conformal transformation generated by this spinor with Q_ϵ . By construction, $Q_\epsilon S_{SYM} = Q_\epsilon W_{\text{circle}} = 0$, moreover Q_ϵ^2 yields a combination of a rotation and of an R -symmetry transformation and thus it generates a compact subgroup of the bosonic symmetries. We now add the following Q_ϵ -exact term

$$V = (\Psi, \bar{Q} \Psi), \quad (2.41)$$

where Ψ is the Majorana-Weyl spinor present in the theory. The corresponding bosonic part of the $(Q_\epsilon V)$ -term,

$$S_{bos}^Q = (Q\Psi, \overline{Q\Psi}). \quad (2.42)$$

is *semipositive* definite. After a tedious computation explicitly performed in [66], one can conclude that the critical points of S_{bos}^Q are given by

$$\phi_0 = 0, \quad K_i = -w_i a \quad (i = 1, 2, 3; \text{ with } w_i w^i = 1/r^2) \quad [\text{other fields}] = 0, \quad (2.43)$$

where a is a matrix belonging to the Lie algebra of the gauge group G and it is a constant over the four sphere. The infinite dimensional path integral thus *localizes* over the finite dimensional locus described by the hermitian matrices a , namely it reduces to the matrix integral

$$\mathbb{Z} \Big|_{\substack{\phi_0=0, \quad K_i=-w_i a \\ [\text{other fields}]=0}} = \int [da] e^{-\frac{4\pi^2 r^2}{g_{YM}^2} \text{Tr}[a^2]} \text{Tr}[e^{2\pi r i a}]. \quad (2.44)$$

To be more precise this is not the all story. We have an additional contribution given by the fluctuation determinants around the locus (2.43), the so-called one-loop determinants for the different fields appearing in the action. In [66] it is shown that the fermionic and bosonic contributions cancel and produce just 1 because of supersymmetry. We refer to the original paper [66] for the details. Therefore (2.44) is the exact value for the circular Wilson loop.

2.5 Wilson loops at strong coupling and Minimal Surface

Now we have a rigorous derivation of the matrix model that captures the *VEV* of the circular Wilson loop for every N and every coupling constant g . As previously discussed it is interesting to verify the matching between the matrix model result in the large λ limit, eq. (2.27), and the *VEV* at strong coupling calculated using the dual supergravity picture.

Recalling that the physical meaning of the Wilson loop (2.4) (on the gauge side) is the phase factor associated to an infinitely massive W-boson (obtained by the higgsing of the gauge group), we can construct the dual string state as follows.

Consider a stack of $N + 1$ D3-branes (giving the $SU(N + 1)$ gauge theory); we place one of them very far away from the others. In this way the gauge group is higgsed to $SU(N) \times U(1)$ and the W-boson that arise is described by an open string stretched in the AdS space from the boundary to the interior.

The Wilson loop can thus be viewed as a surface that stretches in the *AdS* space and that ends on the contour C on which the loop is defined [5, 65, 69]. So a

very natural prescription to compute its VEV is to calculate the string partition function with given boundary conditions.

In general we should consider the full partition function of the string theory on $AdS^5 \times S^5$ but in the large 't Hooft limit it can be computed using a saddle point approximation and is related to the area $A(C)$ of a minimal surface bounded by C

$$\langle W \rangle \simeq e^{-\sqrt{\lambda} A(C)}. \quad (2.45)$$

To be more precise the expectation value $\langle W \rangle$ is not given by the Area $A(C)$ (which is actually infinite) but by an appropriate Legendre transformation [65] since not all the boundary conditions of the open string are of the same type [six are Neumann's and four are Dirichlet's]. The Legendre transformation is performed only for the coordinates obeying Neumann conditions and it is obtained by adding a boundary contribution to the original action. This procedure will also eliminate the divergence present in $A(C)$ and will produce a finite result for the area. From a more physical point of view this transformation corresponds to subtract from the action a divergent factor corresponding to the infinite mass of the W-boson.

For the circular case, the solution corresponding to the minimal area can be easily found if we start with the Polyakov action in the conformal gauge. Since the loop lies on a two dimensional subspace of \mathbb{R}^4 and couples with only one scalar we can neglect the S^5 part of the solution: the open string is frozen in a point of S^5 . Moreover, in AdS_5 , we can look for a minimal surface which lies entirely in a AdS_3 subspace. The metric on this subspace can be written as

$$ds^2 = \frac{L^2}{y^2} (dy^2 + dr^2 + r^2 d\phi^2) \quad (2.46)$$

where (r, ϕ) are the polar coordinates of the plane on which the loop is defined, y is the coordinate transverse to the \mathbb{R}^4 space and L is the radius of AdS_5 . Choosing the static gauge $\phi = \tau$ and making the ansatz

$$y = y(\sigma), \quad r = r(\sigma) \quad (2.47)$$

with (τ, σ) the worldsheet coordinates ($0 < \tau < 2\pi$ and $0 < \sigma < \infty$), the Lagrangian in the conformal gauge can be written as

$$\mathcal{L} = \frac{L^2}{y^2} \left(\left(\frac{dy}{d\sigma} \right)^2 + \left(\frac{dr}{d\sigma} \right)^2 + r^2 \right). \quad (2.48)$$

The solution to the Virasoro constraint

$$\left(\frac{dy}{d\sigma} \right)^2 + \left(\frac{dr}{d\sigma} \right)^2 = r^2 \quad (2.49)$$

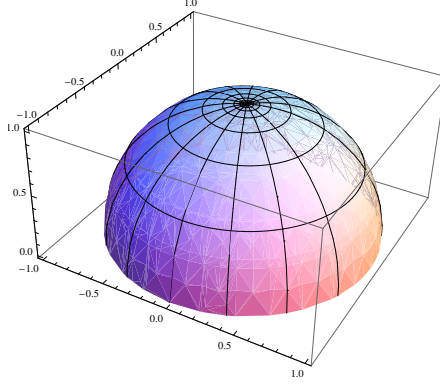


Figure 2.2: Minimal Surface ending on the circular loop

and to the equations of motion

$$y \left(\frac{d^2 y}{d\sigma^2} \right) - \left(\frac{dy}{d\sigma} \right)^2 + \left(\frac{dr}{d\sigma} \right)^2 + r^2 = 0 \quad (2.50)$$

$$\left(\frac{d^2 r}{d\sigma^2} \right) - r - \frac{2}{y} \left(\frac{dy}{d\sigma} \right) \left(\frac{dr}{d\sigma} \right) = 0 \quad (2.51)$$

is given by

$$y = \tanh \sigma \quad r = \frac{1}{\cosh \sigma} \quad (2.52)$$

(see fig.(2.2)). If we evaluate the classical string action on this solution after the inclusion of the boundary term [65]

$$\mathcal{S}_B = \sqrt{\lambda} \int_0^\infty d\sigma \frac{1}{y^2} \left(y \left(\frac{d^2 y}{d\sigma^2} \right) - \left(\frac{dy}{d\sigma} \right)^2 \right) \quad (2.53)$$

we obtain

$$\mathcal{S}_{reg} = \sqrt{\lambda} \int_0^\infty d\sigma \left(\frac{1}{\sinh(\sigma)^2} - \frac{1}{\sinh(\sigma)^2} - \frac{1}{\cosh(\sigma)^2} \right) = -\sqrt{\lambda} \quad (2.54)$$

and so

$$\langle W \rangle \simeq e^{-S} = e^{\sqrt{\lambda}}. \quad (2.55)$$

Quite surprisingly this is essentially the strong coupling expansion of the conjectured matrix model. The factor $\lambda^{-3/4}$ in the eq. (2.27) is recovered by taking into account the zero-modes of the fluctuations around the minimal surface [65].

Chapter 3

Supersymmetric Wilson Loop on S^3

3.1 Supersymmetric properties

In order to study loop operators preserving less supersymmetry than the circle but still exactly solvable, an interesting class of BPS Wilson loops has been considered in [58, 59, 60]. These loops lie on a three-sphere and couple to only three of the six scalars of the $\mathcal{N} = 4$ SYM. They can be written as

$$W = \frac{1}{N} \text{Tr} \mathcal{P} \exp \oint dx^\mu (iA_\mu - \sigma_{\mu\nu}^i x^\nu M^i_I \Phi^I) \quad (3.1)$$

where the matrix M^i_I is 3×6 dimensional the $\sigma_{\mu\nu}^i$ are basically the 't Hooft symbol used in writing down the instanton solution and they are defined in terms of the right-invariant one-form σ_i^R on the three sphere:

$$\sigma_1^R = 2[(x^2 dx^3 - x^3 dx^2) + (x^4 dx^1 - x^1 dx^4)] = 2\sigma_{\mu\nu}^1 x^\mu dx^\nu \quad (3.2a)$$

$$\sigma_2^R = 2[(x^3 dx^1 - x^1 dx^3) + (x^4 dx^2 - x^2 dx^4)] = 2\sigma_{\mu\nu}^2 x^\mu dx^\nu \quad (3.2b)$$

$$\sigma_3^R = 2[(x^1 dx^2 - x^2 dx^1) + (x^4 dx^3 - x^3 dx^4)] = 2\sigma_{\mu\nu}^3 x^\mu dx^\nu. \quad (3.2c)$$

Local supersymmetry (2.9) imposes that MM^T is the unit 3×3 matrix.

The basic idea behind this family of Wilson loops is a topological twist. The particular choice of the scalar couplings breaks the R -symmetry group $SU(4)_R$ of the $\mathcal{N} = 4$ SYM theory to a $SU(2)_a \times SU(2)_b$ where the former group rotates the scalars that couple with the loop while the latter the remaining ones. The twist consists into replacing the anti-chiral part of the Lorentz group $SU(2)_R$ with the diagonal part of $SU(2)_R$ and $SU(2)_a$. After this replacement two of the supercharges are no longer anticommuting spinors, but become anticommuting scalars and close on a

R -symmetry transformation (type II in the Vafa-Witten classification [70]). Then one can regard this two scalars supercharges as BRST charges and the Wilson loop will be an observable in their cohomology.

In the following we review the supersymmetric properties for a subclass of these operators, namely when the contour lies on a two sphere inside S^3 and we list the results for more general circuits into table 1.

The invariant one-forms on S^2 can be easily written as $\sigma_i^R = 2\epsilon_{ijk}x^j dx^k$ and thus the Wilson loops (3.1) for a generic shape on the two sphere read

$$W = \frac{1}{N} \text{Tr} \mathcal{P} \exp \oint ds (iA_\mu x^\mu + \epsilon_{\mu\nu\rho} \dot{x}^\mu x^\nu \Phi^\rho). \quad (3.3)$$

In terms of the conformal Killing spinor $\epsilon(x) = \epsilon_0 + x^\mu \gamma_\mu \epsilon_1$, the vanishing of the supersymmetry variation of (3.3) can be written in a compact form as

$$i\gamma_{jk}\epsilon_1 = \epsilon_{ijk}\rho^i\gamma^5\epsilon_0 \quad (3.4)$$

where γ_μ are the usual Dirac matrices of $SO(4)$ and ρ^i belong to the $SO(6)$ Clifford algebra (R -symmetry group). Since (3.4) is a set of three independent equations ($jk = \{01\}, \{02\}, \{12\}$) we can conclude that for a generic contour on S^2 the Wilson loop preserves 1/8 of the original supersymmetries. The four supercharges preserved may be written as

$$Q^a = (i\tau_2)_a^\alpha (Q_\alpha^{\dot{a}a} + S_\alpha^{\dot{a}a}) \quad \bar{Q}^a = \epsilon^{\dot{a}a} (\bar{Q}_{\dot{a}a}^a - \bar{S}_{\dot{a}a}^a) \quad (3.5)$$

and they are a linear combination of the Poincaré supersymmetries and of the superconformal ones. Some remarks on the notation are in order. The original $SU(4)$ index I (Q^I) has been splitted into two $SU(2)$ spinor indices (a, \dot{a}) . The dotted index \dot{a} belongs to the fundamental of $SU(2)_a$, while the second one a lives in the fundamental of $SU(2)_b$. The 4 supercharges are collected in two $SU_b(2)$ doublets.

It is also instructive to obtain the full supergroup preserving this subclass of operators. Since the loop lies on a two sphere ($x_4 = 0$) there is an extra $U(1)$ bosonic symmetry generated by

$$L = \frac{1}{2} \mathbb{I}^{\alpha\dot{\alpha}} (P_{\alpha\dot{\alpha}} - K_{\alpha\dot{\alpha}}). \quad (3.6)$$

To write the algebra satisfied by these charges, it is convenient to consider the linear combinations

$$Q_\pm^a = \frac{1}{2}(Q^a \pm \bar{Q}^a), \quad (3.7)$$

which are the eigenstates of L . We obtain the superalgebra

$$[Q_+^a, Q_+^b]_+ = [Q_-^a, Q_-^b]_+ = 0 \quad [Q_+^a, Q_-^b]_+ = -T^{ab} + \epsilon^{ab} L$$

$$[L, Q_\pm^a] = \pm Q_\pm^a \quad .$$

Here T^{ab} are the generators of $SU_b(2)$ in spinor notation and while the commutators with themselves are canonical. The above superalgebra is isomorphic to $SU(1|2)$.

For particular shapes of the loop inside S^2 the supersymmetry preserved is enhanced. For example if the circuit is a latitude at polar angle θ_0 the Wilson loop is 1/4 BPS. Indeed parameterizing the loop as

$$x^\mu = (\sin \theta_0 \cos t, \sin \theta_0 \sin t, \cos \theta_0, 0) \quad (3.8)$$

the supersymmetry variation is given in terms of two independent constraints (if $\theta_0 \neq \pi/2$ otherwise only one)

$$\cos \theta_0 (\gamma_{12} + \rho_{12}) \epsilon_1 = 0 \quad (3.9)$$

$$\rho^3 \gamma^5 \epsilon_0 = [i \gamma_{12} + \gamma_3 \rho^2 \gamma^5 \cos \theta (\gamma_{23} + \rho_{23})] \epsilon_1. \quad (3.10)$$

The eight supercharges preserved by the latitude are the four one in (3.5) plus

$$\begin{aligned} Q^a &= \frac{1}{\sin \theta_0} (\tau_3 \epsilon)_\dot{a}^\dot{a} (\bar{Q}_{\dot{a}a}^a - \bar{S}_{\dot{a}a}^a) + \cot \theta_0 (i \tau_2)_\alpha^{\dot{a}} (Q_{\dot{a}a}^{\dot{a}a} - S_{\dot{a}a}^{\dot{a}a}) \\ Q^a &= \frac{1}{\sin \theta_0} (\tau_1)_\dot{a}^\dot{a} (Q_{\dot{a}a}^{\dot{a}a} + S_{\dot{a}a}^{\dot{a}a}) + \cot \theta_0 (\epsilon)^{\dot{a}a} (\bar{Q}_{\dot{a}a}^a + \bar{S}_{\dot{a}a}^a). \end{aligned} \quad (3.11)$$

Interestingly one can note, from the explicit expression of the scalar couplings, that this configuration describes also a latitude at angle $\pi/2 - \theta_0$ on the dual $S^2 \subset S^5$ associated to the three scalar.

The last configuration that we analyze is 1/4 BPS and it can be obtained considering two arcs of length π connected at an arbitrary angle δ . The explicitly parametrization is

$$x = \begin{cases} (\sin t, 0, \cos t, 0) & 0 \leq t \leq \pi \\ (-\cos \delta \sin t, -\sin \delta \sin t, \cos t, 0) & \pi \leq t \leq 2\pi \end{cases} \quad .$$

The operator couples with ϕ_1 on the first arc of the circuit and with $-\phi_1 \cos \delta + \phi_0 \sin \delta$ on the second one (on the dual S^2 this configuration corresponds to two points separated by an angle $\pi - \delta$). The supersymmetric variation of this operator is given by two independent equations

$$\rho_2 \gamma_5 \epsilon_0 = i \gamma_{31} \epsilon_1 \quad (3.12)$$

$$\sin \delta \rho_1 \gamma_5 \epsilon_0 = i \sin \delta \gamma_{23} \epsilon_1 \quad (3.13)$$

(if $\sin \delta \neq 0$ otherwise only one) and thus the supercharges preserve are eights and read as

$$\begin{aligned}
Q_1^a &= (i\tau_2)_\alpha^\alpha (Q_\alpha^{aa} + S_\alpha^{aa}) & \bar{Q}_1^a &= \epsilon^{\dot{\alpha}\dot{\alpha}} (\bar{Q}_{\dot{\alpha}\dot{\alpha}}^a - \bar{S}_{\dot{\alpha}\dot{\alpha}}^a) \\
Q_2^a &= (\tau_1)_\alpha^\alpha (Q_\alpha^{aa} - S_\alpha^{aa}) & \bar{Q}_2^a &= (\tau_3\epsilon)^{\dot{\alpha}\dot{\alpha}} (\bar{Q}_{\dot{\alpha}\dot{\alpha}}^a + \bar{S}_{\dot{\alpha}\dot{\alpha}}^a).
\end{aligned}$$

Other configurations with different amount of supersymmetries and the respective superalgebras preserved are listed in the table below.

Circuits	Supercharges preserved	Superalgebra
Equator on S^2	1/2 BPS	$OSp(4 4)$
General Curves on S^2	1/8 BPS	$SU(1 2)$
Latitude on S^2	1/4 BPS	$SU(2 2)$
Two Longitudes on S^2	1/4 BPS	$SU(1 2) \times SU(1 2)$
Hopf fiber on S^3	1/8 BPS	$OSp(1 2) \times OSp(1 2)$

Table n. 1: Supersymmetries of different loops

3.2 From $\mathcal{N} = 4$ SYM in $4d$ to YM_2 on S^2

In the present subsection we show some evidences about the equivalence between the above class of Wilson loops on S^2 and the analogous observables in the zero-instanton sector of the ordinary bosonic two dimensional Yang Mills on S^2 [59, 60]. The YM_2 is a very well-known theory (see *appendix B*) and it has a series of interesting properties. It is almost topological, namely it is invariant under area preserving diffeomorphism, and it is exactly solvable. It would be nice to relate it to the more complex four dimensional $\mathcal{N} = 4$ SYM theory.

We begin considering a (DGRT) Wilson loop defined on a generic path in S^2 and we try to calculate its perturbative VEV at first order in the coupling constant. This operator has the form

$$W = \frac{1}{N} \text{Tr } \mathcal{P} \exp \left(\oint ds (iA_\mu \dot{x}^\mu + \epsilon_{\mu\nu\rho} \dot{x}^\mu \dot{x}^\nu \Phi^\rho) \right) \quad (3.14)$$

and a simple evaluation of its expectation value up to g_{4d}^2 order gives

$$\begin{aligned}
\langle W \rangle &= 1 - \frac{1}{2N} \text{Tr } \mathcal{P} \int dx^i dy^i \left(\langle A_i(x) A_j(y) \rangle - \epsilon_{ikl} \epsilon_{jmn} x^k y^m \langle \phi^l(x) \phi^n(y) \rangle \right) = \\
&= 1 - \frac{g_{4d}^2 N}{8\pi^2} \oint_{s>t} ds dt \dot{x}^i(s) \dot{y}^j(t) \left(\frac{1}{2} g_{ij} - \frac{(x-y)_i (x-y)_j}{(x-y)^2} \right) = \\
&= 1 - \frac{g_{4d}^2 N}{16\pi^2} \oint ds dt \dot{x}^i(s) \dot{y}^j(t) \left(\frac{1}{2} g_{ij} - \frac{(x-y)_i (x-y)_j}{(x-y)^2} \right). \quad (3.15)
\end{aligned}$$

The last line can be evaluated with the help of the Stokes theorem applied, for example, to the y -contour integration and subsequently to x -contour. One obtains

$$W = 1 + \frac{g_{4d}^2 N}{4\pi} \frac{\mathcal{A}_1 \mathcal{A}_2}{2\mathcal{A}} + \mathcal{O}(g_{4d}^4), \quad (3.16)$$

where \mathcal{A}_1 and \mathcal{A}_2 are the areas delimited by the loops and $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ is the total area of the two sphere. Intriguingly we see that the overall result does not depend on the particular shape of the loop, but just on the area of the two sectors $\mathcal{A}_1, \mathcal{A}_2$: it suggests a sort of invariance under area preserving transformations. This fact and the appearance of the peculiar combination $\frac{\mathcal{A}_1 \mathcal{A}_2}{2\mathcal{A}}$ bring to mind a similar result for the pure Yang-Mills theory on the two-dimensional sphere [71].

In that case the theory is completely solvable [72] and the exact expression for the ordinary Wilson loop is available [73, 74]: restricting the full answer to the zero-instanton sector (for details see *appendix B*) one obtains

$$\langle W \rangle = \frac{1}{N} L_{N-1}^1 \left(g_{2d}^2 \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}} \right) \exp \left[-\frac{g_{2d}^2}{2} \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}} \right] \quad (3.17)$$

where $L_{N-1}^1(x)$ is a Laguerre polynomial (2.24). Let us notice that its first order perturbative expansion ($g_{2d} \ll 1$)

$$\langle W \rangle = 1 - g_{2d}^2 N \frac{\mathcal{A}_1 \mathcal{A}_2}{2\mathcal{A}} + g_{2d}^4 (2N^2 + 1) \frac{\mathcal{A}_1^2 \mathcal{A}_2^2}{24\mathcal{A}^2} + \mathcal{O}(g_{2d}^6) \quad (3.18)$$

coincides exactly with the $\mathcal{N} = 4$ SYM result (3.16) after identifying the two-dimensional coupling constant g_{2d}^2 with the four-dimensional one through

$$g_{2d}^2 = -\frac{g_{4d}^2}{4\pi}. \quad (3.19)$$

Generalizing this result the authors of [60] have thus conjectured that the 1/8 BPS Wilson loops constructed on S^2 can be computed exactly leading to the two-dimensional result.

BPS Wilson loops on S^2 in $\mathcal{N} = 4$ SYM
\equiv
Wilson loops on S^2 in the zero instanton sector of YM_2

As we will see in the following this conjecture has passed a series of different and more refined checks. In the sections (3.3) and (3.4) we report a detail two-loop analysis [75, 76] and a strong coupling check [60] of the conjecture finding in both cases an exact agreement between $\mathcal{N} = 4$ SYM calculation and the prediction

(3.17). From these results one could infer that a localization procedure like those presented in [66] and briefly illustrated in section (2.4) for the Maldacena-Wilson loops could also apply to this more general class of operators. Indeed in [77] the author has shown that the 4d path integral localizes to the semi-topological Hitchin/Higgs-Yang-Mills theory and that perturbatively the computation of this class of Wilson loops reduces to an analogous computation in the zero instanton sector of the YM_2 on S^2 . The proof of this reduction however is not complete. There are two missing ingredients: an exact computation of the fluctuation determinant is still missing and it is not clear if the $\mathcal{N} = 4$ instanton contribute.

These relations between the $\mathcal{N} = 4$ SYM in four dimensions and the ordinary bosonic YM_2 has been intensively studied also in [78, 79, 80] where the link has been extended to the case of the correlators of Wilson loops (we analyze in detail this point in the chapter five). The two-matrix model capturing the VEV of the correlators in the two dimensional theory has been calculated and an agreement with the $\mathcal{N} = 4$ SYM calculation has been found both at perturbative and non-perturbative level.

The Correlator of two Wilson loops on S^2 in $\mathcal{N} = 4$ SYM
 \equiv
The Correlator of two Wilson loops on S^2
in the zero instanton sector of YM_2

In another interesting paper [26] the authors have generalized the proposal into another direction: they have conjectured that in $\mathcal{N} = 4$ SYM the insertion of 't Hooft operators on a maximal circle on S^2 and their correlators with the Wilson loops are captured by the non-zero (unstable) instanton contributions to the partition function of the 2d Yang-Mills theory.

The Correlator of 1/2 BPS 't Hooft loop and 1/8 BPS Wilson loop
on S^2 in $\mathcal{N} = 4$ SYM
 \equiv
The Wilson loop on S^2 in a non-zero instanton sector of YM_2

't Hooft loops are disorder operators and, differently from the Wilson loops, there isn't a direct formulation in terms of fields. They are defined by giving a singularity for the fields near the loop C on which the 't Hooft loops are defined [27, 81]. To compute their expectation value one has to integrate over the configurations that are smooth everywhere except on C where they are required to have the prescribed singular behavior. This singularity is the one associated to the 1/2 BPS monopole

configurations where the field strength and scalars behave [81] as

$$\begin{aligned} F_{jk}(y) &= \frac{1}{2} \epsilon_{ijk} \frac{y_i}{|y|^3} T_{\vec{m}} \\ \phi_0(y) &= \frac{T_{\vec{m}}}{2} \frac{1}{|y|} \\ \phi_i(y) &= 0 \quad (i = 1 \dots 5) \end{aligned} \tag{3.20}$$

where \vec{m} is a set of N integers, $T_{\vec{m}}$ is a $U(N)$ diagonal matrix with entries given by \vec{m} and $|y|$ is the distance from the loop. The authors note that when one restricts this configuration to the boundary sphere S^2 , it becomes precisely of the same form of the unstable-instanton contributions to YM_2 , the classical configurations solving $DF = 0$ on which the partition function of 2d YM on S^2 localizes [82, 83] (see *appendix B*). This new conjecture has not yet been proved but some indications about its validity are shown in the original paper [26]. Nonetheless it would be nice to obtain other non-trivial checks (for example an agreement between the matrix model and the $\mathcal{N}=4$ SYM computation of the 't Hooft/Wilson loop correlators).

3.3 Two-Loop check of the conjecture

Two-loop expansion for supersymmetric loops on S^2

To verify the conjecture illustrated in the previous subsection, a non-trivial two-loop computation is in order. Thus in the following we discuss this perturbative expansion for a supersymmetric Wilson loops lying on S^2 . To perform a quantum analysis we will need to adopt a regularization procedure since, as we will see, divergent diagrams could appear in the intermediate steps of the computations. We choose the familiar dimensional reduction, consisting in considering $\mathcal{N} = 4$ SYM in 2ω dimensions as a dimensional reduction of $\mathcal{N} = 1$ SYM in ten dimensions. We shall perform firstly our analysis for a generic contour and subsequently we shall consider and calculate numerically the specific example of spherical wedge whose boundary is two longitudes of the sphere S^2 .

The first ingredient in our computations is the effective gluon-scalar propagator appearing in the perturbative expansion of the Wilson loop (3.14). In 2ω dimensions and for a generic circuits on S^2 it has the simple form

$$\Delta^{ab}(t_1, t_2) = \delta^{ab} \frac{\Gamma(\omega - 1)}{4\pi^\omega} \frac{(\dot{x}_1 \cdot \dot{x}_2)[(x_1 \cdot x_2) - 1] - (x_1 \cdot \dot{x}_2)(x_2 \cdot \dot{x}_1)}{((x_1 - x_2)^2)^{\omega-1}}. \tag{3.21}$$

In order to investigate the singular behavior of the supersymmetric Wilson loop, it is instructive to study the effective propagator when t_1 approaches t_2 , *i.e.* when

the two points on the contour are about to collide: it is convenient to rearrange $\Delta^{ab}(t_1, t_2)$ as follows

$$\Delta^{ab}(t_1, t_2) = \delta^{ab} \frac{\Gamma(\omega - 1)}{4\pi^\omega} \left[\frac{1}{2} (\dot{x}_1 \cdot \dot{x}_2) ((x_1 - x_2)^2)^{2-\omega} + \frac{(x_1 - x_2) \cdot \dot{x}_2 (x_1 - x_2) \cdot \dot{x}_1}{((x_1 - x_2)^2)^{\omega-1}} \right]. \quad (3.22)$$

Let us consider the case of smooth loops: the above expression (3.22) is composed by two contributions, that are of the same order in the coincidence limit. A straightforward Taylor-expansion gives indeed for (3.22) the following leading behavior $\Gamma(\omega - 1)(|\dot{x}_1|^2)^{3-\omega}(t_1 - t_2)^{4-2\omega}$ and it is completely finite when $1 < \omega \leq 2$. An analogous result holds for smooth loops with the usual constant coupling, where Θ^I is a constant unit vector in \mathbb{R}^6 , as first shown in [64]. In this last case divergencies at coinciding points could appear when considering loops endowed with cusps and are related to the famous "cusp anomaly". We can examine the non-smooth loop in our case as well: here the situation is more subtle. Let x_1 and x_2 be the extreme of the propagator approaching the cusp from the left and the right respectively. If the cusp is located at $x = x_0$, we can always choose the parametrization of the contour such that $x_1 = x_0 + t_1 n_1 + O(t_1^2)$ and $x_2 = x_0 + t_2 n_2 + O(t_2^2)$ with $n_1^2 = n_2^2 = 1$ and $t_1, t_2 \geq 0$. Then we find that the leading behavior of $\Delta^{ab}(t_1, t_2)$ when both t_1 and t_2 are close to zero is given just by the second term

$$\Delta^{ab}(t_1, t_2) \sim \delta^{ab} \frac{\Gamma(\omega - 1)}{4\pi^\omega} \frac{2(1 - (n_1 \cdot n_2)^2)t_1 t_2}{(t_1^2 + t_2^2 - 2t_1 t_2 (n_1 \cdot n_2))^{\omega-1}}. \quad (3.23)$$

A simple power counting argument shows that this object is integrable around $(t_1, t_2) = (0, 0)$ for all the values of ω less than 3 and in particular for $\omega = 2$. This regular behavior entails therefore an important difference between the family of loops with the new scalar couplings and the ones previously considered: no singular contribution is associated here to the presence of the cusp. In some way the celebrated cusp anomaly appears to be smoothed away at the leading order when considering this class of supersymmetric Wilson loops. Actually we shall see that this also occurs at two loops and probably it is true at all orders.

Then we shall consider the effect of the one loop correction to the effective propagator (3.21). The relevant diagrams are schematically displayed in fig.(3.1) (*bubble diagrams*). The value of the contribution in Feynman gauge can be easily computed with the help of [64], where the one-loop correction to the gauge and scalar propagator has been calculated. The final result is

$$\begin{aligned} S_2 &= -g^4 (N^2 - 1) \frac{\Gamma^2(\omega - 1)}{2^7 \pi^{2\omega} (2 - \omega)(2\omega - 3)} \times \\ &\times \oint d\tau_1 d\tau_2 \frac{(\dot{x}_1 \cdot \dot{x}_2)[(x_1 \cdot x_2) - 1] - (x_1 \cdot \dot{x}_2)(x_2 \cdot \dot{x}_1)}{[(x^{(1)} - x^{(2)})^2]^{2\omega-3}} \equiv \\ &\equiv -g^4 (N^2 - 1) \frac{\Gamma^2(\omega - 1)}{2^7 \pi^{2\omega} (2 - \omega)(2\omega - 3)} \Sigma_\omega[C]. \end{aligned} \quad (3.24)$$

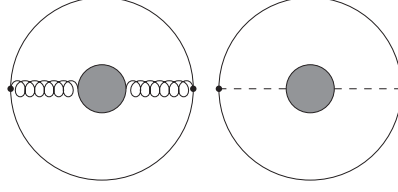


Figure 3.1: One-loop correction to the gluon and the scalar exchange.

with

$$\Sigma_\omega[C] \equiv \oint d\tau_1 d\tau_2 \frac{(\dot{x}_1 \cdot \dot{x}_2)[(x_1 \cdot x_2) - 1] - (x_1 \cdot \dot{x}_2)(x_2 \cdot \dot{x}_1)}{[(x^{(1)} - x^{(2)})^2]^{2\omega-3}}. \quad (3.25)$$

The coefficient of (3.24) exhibits a pole in $\omega = 2$, which keeps track of the divergence in the loop integration.

The next step, at this order, is to investigate the so-called *spider diagrams*, namely the perturbative contributions coming from the gauge vertex A^3 and the scalar-gauge vertex $\phi^2 A$ (see fig. 3.2). We have to compute

$$S_3 = \frac{g^3}{3N} \oint dt_1 dt_2 dt_3 \eta(t_1 t_2 t_3) \langle \text{Tr}[\mathcal{A}(t_1) \mathcal{A}(t_2) \mathcal{A}(t_3)] \rangle_0, \quad (3.26)$$

where the short notation $\mathcal{A}(t_\ell)$ stands for the relevant combination

$$iA_\mu^a(x_\ell) \dot{x}_\ell^\mu - \sigma_{\mu\nu}^s \dot{x}_\ell^\mu \dot{x}_\ell^\nu M^s{}_I \Phi_I^a(x_\ell)$$

and

$$\eta(t_1, t_2, t_3) = \theta(t_1 - t_2) \theta(t_2 - t_3) + \text{cyclic permutations}. \quad (3.27)$$

After a simple, but tedious computation, in Feynman gauge S_3 takes the form

$$S_3 = \frac{g^4(N^2 - 1)}{4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) \times \\ \times [(\dot{x}_1 \cdot \dot{x}_3)[(x_1 \cdot x_3) - 1] - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)] \dot{x}_2^\mu \frac{\partial \mathcal{I}_1(x_3 - x_1, x_2 - x_1)}{\partial x_3^\mu}, \quad (3.28)$$

where we have introduced the symbol

$$\epsilon(t_1, t_2, t_3) = \eta(t_1, t_2, t_3) - \eta(t_2, t_1, t_3),$$

that is a totally antisymmetric object in the permutations of (t_1, t_2, t_3) and its value is 1 when $t_1 > t_2 > t_3$. The quantity $\mathcal{I}_1(x, y)$ ¹ is defined as the following integral in momentum space

$$\mathcal{I}_1(x, y) \equiv \int \frac{d^{2\omega} p_1 d^{2\omega} p_2}{(2\pi)^{4\omega}} \frac{e^{ip_1 x + ip_2 y}}{p_1^2 p_2^2 (p_1 + p_2)^2}. \quad (3.29)$$

¹ $\mathcal{I}_1(x, y)$ is evaluated and its properties are discussed in detail in *appendix C*.

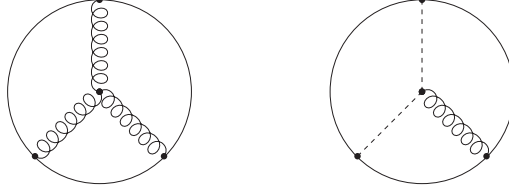


Figure 3.2: Spider-diagrams: gauge and scalar contribution

It is quite important, at this point, to understand the potential divergences arising in (3.28) when $\omega \rightarrow 2$: their appearance originates directly from the integration over the contour. In fact since the integral (3.29) is finite and regular for $x, y \neq 0$, singularities can only arise in the contour integration when two of the x_i collide. In that case, a pole at $\omega = 2$ appears in the expression of $\mathcal{I}_1(x, y)$: for $x_1 = x_2$ one finds (see *appendix C*)

$$\mathcal{I}_1(x_3 - x_1, 0) = \frac{\Gamma^2(\omega - 1)}{(2\omega - 3)(2 - \omega)} \frac{1}{64\pi^{2\omega} [(x_1 - x_3)^2]^{2\omega-3}}. \quad (3.30)$$

The same behavior occurs when $x_1 = x_3$ or $x_2 = x_3$ since \mathcal{I}_1 is totally symmetric in the exchange of the x_i : we observe three different regions, namely $[(x_1 \simeq x_2), (x_1 \simeq x_3), (x_2 \simeq x_3)]$, which are potential sources of divergences. Actually, the situation is better than what one would naively expect: the true singularity at $\omega \rightarrow 2$ appears just in a single region, for the following reasons. One observes that the divergent behavior at $x_1 = x_3$ becomes integrable because of the presence of the kinematical pre-factor $(\dot{x}_1 \cdot \dot{x}_3)[(x_1 \cdot x_3) - 1] - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)$, inherited by the vector/scalar coupling, which nicely vanishes in this limit. The contribution coming from the region $x_1 \simeq x_2$ becomes instead ineffective due to the derivative with respect to x_3 , when acting on \mathcal{I}_1 . The only dangerous singularity appears when x_3 approaches x_2 .

A similar pattern for the divergences was discussed in [64] for the usual Wilson-Maldacena loop (the loop with constant θ^I). The authors made the crucial observation that the residual divergence at $x_2 \simeq x_3$ is exactly compensated by a contribution coming from the one-loop correction to the *effective* propagator. A subtle cancelation among singularities in the contour integration and the loop integration yields a completely finite result for the Wilson-Maldacena loop at the fourth-order in perturbation theory, in the case of smooth circuits. This nice conclusion suggests that the divergences appearing in each diagram are indeed gauge artefacts and do not have a physical meaning, canceling out in the final result. In particular, one could expect that all the diagrams can be made separately finite with a suitable choice of gauge: in *appendix A* of [75], as an example, it is shown that the light-cone gauge does enjoy this property for smooth circuits lying in the plane orthogonal to light-cone directions.

The situation is analogous for the class of supersymmetric Wilson-loop we are considering. Firstly, we shall show that we can explicitly factor out the divergent part of the *spider diagram* and that it has the same form of the *bubble* contribution. This can be achieved by rearranging the original expression (3.28) for S_3 with the help of this trivial identity:

$$0 = \frac{g^4(N^2 - 1)}{4} \oint dt_1 dt_2 dt_3 \frac{d}{dt_2} \left[\epsilon(t_1, t_2, t_3) ((\dot{x}_1 \cdot \dot{x}_3)((x_1 \cdot x_3) - 1) - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)) \mathcal{I}_2(x_3 - x_2, x_1 - x_2) \right]. \quad (3.31)$$

The definition and the properties of the function \mathcal{I}_2 are listed in *appendix C*. With this addition, the expression can be rearranged by decomposing S_3 as the sum of two different contributions, $S_3 = \mathcal{A} + \mathcal{B}$, as follows

$$S_3 = \frac{g^4(N^2 - 1)}{4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) [(\dot{x}_1 \cdot \dot{x}_3)[(x_1 \cdot x_3) - 1] - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)] \times \left\{ (\mathcal{A}) - \times \dot{x}_2^\mu \left[\frac{\partial \mathcal{I}_1(x_3 - x_1, x_2 - x_1)}{\partial x_3^\mu} - \frac{\partial \mathcal{I}_2(x_3 - x_1, x_2 - x_1)}{\partial x_2^\mu} \right] \right\} - \frac{g^4(N^2 - 1)}{2} \oint dt_1 dt_3 ((\dot{x}_1 \cdot \dot{x}_3)((x_1 \cdot x_3) - 1) - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)) \times \left\{ (\mathcal{B}) \times [\mathcal{I}_2(x_3 - x_1, x_3 - x_1) - \mathcal{I}_2(x_3 - x_1, 0)] \right\}, \quad (3.32)$$

where we have used

$$\frac{d}{dt_2} \epsilon(t_1, t_2, t_3) = 2(\delta(t_2 - t_3) - \delta(t_1 - t_2)). \quad (3.33)$$

We start by focussing our attention on \mathcal{B} : it has exactly the same structure of the result S_2 , produced by the bubble diagrams, as can be easily inferred looking at the kinematical prefactor. We are led to collect all these contributions and sum them together. Exploiting the explicit behavior of $\mathcal{I}_2(x, y)$ for $y = x$ and $y = 0$, as given in *appendix C*, we can write the sum of all *bubble-like* contributions \mathcal{B}_{tot} as

$$\begin{aligned} \mathcal{B}_{\text{tot}} = S_2 + \mathcal{B} &= \frac{g^4(N^2 - 1)}{128\pi^{2\omega-1} \sin \pi\omega} \left(\frac{\Gamma(\omega - 2)}{\Gamma(3 - \omega)} - 2\Gamma(2\omega - 4) \right) \Sigma_\omega[C] = \\ &= - \frac{g^4(N^2 - 1)}{384\pi^2} \Sigma_2[C] + O((\omega - 2)). \end{aligned} \quad (3.34)$$

In other words, when we sum the term \mathcal{B} present in (3.32) to the original one-loop correction coming from the *bubble diagrams*, we obtain a completely finite result \mathcal{B}_{tot} , where the pole in $\omega = 2$ has disappeared. Since the contour integration is also finite in this limit, we can consistently pose $\omega = 2$ in (3.34).

The finiteness of (3.34) clearly hints that also the combination \mathcal{A} appearing in (3.32) is free of divergencies, as ω approaches two: this is indeed the case. The

contribution \mathcal{A} can be rewritten as follows, once the derivatives have been explicitly taken

$$\begin{aligned} \mathcal{A} = & -\frac{g^4(N^2-1)\Gamma(2\omega-2)}{128\pi^{2\omega}(\omega-1)} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) [(\dot{x}_3 \cdot \dot{x}_1)[(x_1 \cdot x_3) - 1] - \\ & - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)] \int_0^1 d\alpha \frac{[\alpha(1-\alpha)]^{\omega-1} \dot{x}_2 \cdot (x_1 - x_3)}{(\alpha(x_3 - x_2)^2 + (1-\alpha)(x_2 - x_1)^2)^{2\omega-2}} {}_2F_1(1, 2\omega-2; \omega; \xi). \end{aligned} \quad (3.35)$$

Here we have denoted with ξ the following combination of the original coordinates

$$\frac{(x_3 - x_1 - \alpha(x_2 - x_1))^2}{\alpha(x_3 - x_2)^2 + (1-\alpha)(x_2 - x_1)^2},$$

which appears in the argument of the hypergeometric function ${}_2F_1(1, 2\omega-2; \omega; \xi)$. The integral (3.35) is nicely convergent in the limit $\omega \rightarrow 2$, in fact by setting $\omega = 2$ we find

$$\begin{aligned} \mathcal{A} = & \frac{g^4(N^2-1)}{128\pi^4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) \frac{(\dot{x}_1 \cdot \dot{x}_3)[(x_1 \cdot x_3) - 1] - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)}{(x_3 - x_1)^2} \times \\ & \times \dot{x}_2 \cdot (x_3 - x_1) \int_0^1 d\alpha \frac{1}{\alpha(x_3 - x_2)^2 + (1-\alpha)(x_2 - x_1)^2} = \\ = & -\frac{g^4(N^2-1)}{128\pi^4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) \frac{(\dot{x}_1 \cdot \dot{x}_3)[(x_1 \cdot x_3) - 1] - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)}{(x_3 - x_1)^2} \times \\ & \times \frac{\dot{x}_2 \cdot (x_3 - x_1)}{(x_3 - x_2)^2 - (x_2 - x_1)^2} \log \left(\frac{(x_2 - x_1)^2}{(x_3 - x_2)^2} \right). \end{aligned} \quad (3.36)$$

We remark that the original power-like singularity for $x_2 \rightarrow x_3$ has disappeared and it has been replaced by a milder logarithmic one, which is integrable both for smooth and cusped loops.

We can actually go further and extract from (3.36) another bubble-like contribution that cancels completely \mathcal{B}_{tot} ! With the help of the following identity

$$\begin{aligned} \frac{(x_3 - x_1) \cdot \dot{x}_2}{(x_3 - x_2)^2 - (x_1 - x_2)^2} \log \left(\frac{(x_2 - x_1)^2}{(x_3 - x_2)^2} \right) = & \frac{1}{2} \frac{d}{dt_2} \left[\text{Li}_2 \left(1 - \frac{(x_2 - x_1)^2}{(x_3 - x_2)^2} \right) + \right. \\ & \left. + \frac{1}{2} \left(\log \left[\frac{(x_3 - x_2)^2}{(x_3 - x_1)^2} \right] \right)^2 \right] + \frac{(x_3 - x_2) \cdot \dot{x}_2}{(x_3 - x_2)^2} \log \left(\frac{(x_2 - x_1)^2}{(x_3 - x_1)^2} \right), \end{aligned} \quad (3.37)$$

we can integrate by part (3.36). We arrive to the following expression

$$\begin{aligned} \mathcal{A} = & \frac{g^4(N^2 - 1)}{384\pi^2} \Sigma_2[C] + \\ & + \frac{g^4(N^2 - 1)}{128\pi^4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) \frac{(\dot{x}_1 \cdot \dot{x}_3)[(x_1 \cdot x_3) - 1] - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)}{(x_3 - x_1)^2} \times \\ & \times \frac{(x_3 - x_2) \cdot \dot{x}_2}{(x_3 - x_2)^2} \log \left(\frac{(x_2 - x_1)^2}{(x_3 - x_1)^2} \right). \end{aligned} \quad (3.38)$$

We see that the first term exactly cancels \mathcal{B}_{tot} and the only surviving contribution from the spider and the bubble diagrams can be written as a relatively simple convergent integral

$$\begin{aligned} \mathcal{I}_{tot} = & \frac{g^4(N^2 - 1)}{128\pi^4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) \frac{(\dot{x}_1 \cdot \dot{x}_3)[(x_1 \cdot x_3) - 1] - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)}{(x_3 - x_1)^2} \times \\ & \times \frac{(x_3 - x_2) \cdot \dot{x}_2}{(x_3 - x_2)^2} \log \left(\frac{(x_2 - x_1)^2}{(x_3 - x_1)^2} \right). \end{aligned} \quad (3.39)$$

It is remarkable that this expression holds for any kind of loop on S^2 , both cusped and smooth, and being free of divergencies is amenable, if necessary, to a plain numerical evaluation, once the contour is specified. In the circular case \mathcal{I}_{tot} is easily seen to vanish by simple symmetry arguments, recovering without tears the result of [64].

This is not of course the end of story: we have still to consider the *double-exchange* diagrams to the perturbative expansion of the Wilson loop, namely we have to analyze the contribution

$$\frac{g^4}{N} \oint_C dt_1 dt_2 dt_3 dt_4 \theta(t_1 - t_2) \theta(t_2 - t_3) \theta(t_3 - t_4) \langle \text{Tr}[\mathcal{A}(t_1) \mathcal{A}(t_2) \mathcal{A}(t_3) \mathcal{A}(t_4)] \rangle_0. \quad (3.40)$$

Recalling that the effective propagator has the color structure $\Delta^{ab}(t_1, t_2) = \delta^{ab} \Delta(t_1, t_2)$, the relevant Green function can be written as

$$\begin{aligned} \langle \text{Tr}[\mathcal{A}(t_1) \mathcal{A}(t_2) \mathcal{A}(t_3) \mathcal{A}(t_4)] \rangle_0 = & \frac{1}{2} \text{Tr}([T^b, T^a][T^b, T^a]) \Delta(t_1, t_3) \Delta(t_2, t_4) + \\ & + \text{Tr}(T^a T^a T^b T^b) [\Delta(t_1, t_2) \Delta(t_3, t_4) + \Delta(t_1, t_3) \Delta(t_2, t_4) + \Delta(t_1, t_4) \Delta(t_2, t_3)]. \end{aligned} \quad (3.41)$$

The term multiplying $\text{Tr}(T^a T^a T^b T^b)$ is symmetric in the exchange of all the t_i and therefore is insensitive to the path-ordering. It simply yields 1/2 the square of the *single-exchange* contribution

$$\frac{1}{2} \left(\frac{g^2 N}{8\pi^2} \oint_C dt_1 dt_2 \frac{(\dot{x}_1 \cdot \dot{x}_2)[(x_1 \cdot x_2) - 1] - (x_1 \cdot \dot{x}_2)(x_2 \cdot \dot{x}_1)}{(x_1 - x_2)^2} \right)^2. \quad (3.42)$$

This simple manipulation expresses the trivial exponentiation of the so-called abelian part of the Wilson loop. The remaining contribution, which is proportional to $\text{Tr}([T^b, T^a][T^b, T^a])$, is usually called the *maximally non-abelian part* and it is the new ingredient in the *double-exchange* contribution. We are left to compute the integral

$$\begin{aligned}
& - \frac{g^4(N^2 - 1)\Gamma^2(\omega - 1)}{64\pi^{2\omega}} \oint_C dt_1 dt_2 dt_3 dt_4 \theta(t_1 - t_2) \theta(t_2 - t_3) \theta(t_3 - t_4) \times \\
& \times \frac{(\dot{x}_1 \cdot \dot{x}_3)[(x_1 \cdot x_3) - 1] - (x_1 \cdot \dot{x}_3)(x_3 \cdot \dot{x}_1)}{((x_1 - x_3)^2)^{\omega-1}} \times \frac{(\dot{x}_4 \cdot \dot{x}_2)[(x_4 \cdot x_2) - 1] - (x_4 \cdot \dot{x}_2)(x_2 \cdot \dot{x}_4)}{((x_4 - x_2)^2)^{\omega-1}}.
\end{aligned} \tag{3.43}$$

This contribution is of course finite and we can set safely again $\omega = 2$.

The cusped loop on S^2

In the present subsection we will provide a fourth-order evidence that the supersymmetric Wilson loops lying on S^2 are actually equivalent to the usual, non-supersymmetric Wilson loops of Yang-Mills theory on a 2-sphere as conjectured in [60] on the basis of a one-loop calculation (see previous subsections). We have not been able to show this equivalence in general: one should compute (3.39) and (3.43) for a generic contour on S^2 and compare the total result with (3.17). This task seems particularly difficult, especially because we do not see any simple way in which (3.39) and (3.43) could generate something proportional to $(\Sigma_2[C])^2$. In the parent circular case, that corresponds to a contour winding the equator of S^2 , two obvious simplifications appear: the vanishing of (3.39) and the constant behavior of the effective propagator, that allows an easy computation of (3.43). For a generic loop on S^2 both properties seem to disappear, at least in Feynman gauge, and the matrix model result could be recovered only through a delicate interplay among interacting and double-exchange contributions. We are led therefore to check, as first instance, the conjecture against a particular class of loops, for which the calculation of (3.39) and (3.43) is relatively easy.

We will focus on a particular family of 1/4 BPS Wilson loops made of two arcs of length π connected at an arbitrary angle δ (see section 3.1) : another explicit parametrization is

$$x(t) = \begin{cases} \left(-\frac{2t}{1+t^2}, 0, \frac{1-t^2}{1+t^2} \right) & \text{for } -\infty < t \leq 0 \\ \left(\frac{2t}{1+t^2} \cos \delta, \frac{2t}{1+t^2} \sin \delta, \frac{1-t^2}{1+t^2} \right) & \text{for } 0 \leq t < \infty \end{cases} \tag{3.44}$$

This path starts from the south pole of the sphere $(0, 0, -1)$ for $(t = -\infty)$. When t increases, we move along a meridian $\phi = 0$ up to the north pole $(0, 0, 1)$, which is reached for $(t = 0)$. From the north pole, we move back to the south pole along

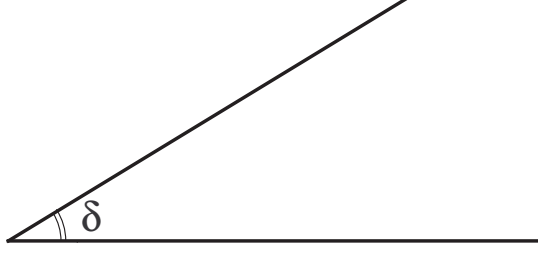
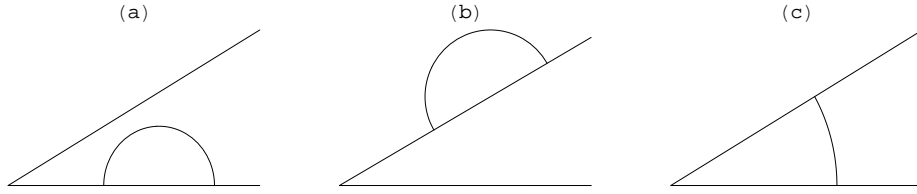


Figure 3.3: Stereographic Projection of the “wedge”.

Figure 3.4: *Single-exchange* diagrams

the meridian $\phi = \delta$ and we again reach the south pole when $t = +\infty$. Notice that this parametrization for the contour is nothing else but its stereographic projection on the plane.

Let us start by discussing, as a warm up, the lowest order contribution. For this kind of loop the *single exchange* splits in three sub-diagrams: the diagram (a) and (b) are equal, since we cannot distinguish the two longitudes. We have that

$$(a) + (b) = 2(a) = \frac{g^2 N}{2\pi^2} \int_{-\infty}^0 dt_1 \int_{-\infty}^{t_1} dt_2 \frac{1}{(t_1^2 + 1)(t_2^2 + 1)} = \frac{g^2 N}{16}. \quad (3.45)$$

The diagram (c) is given by

$$(c) = -\frac{g^2 N}{4\pi^2} \int_0^\infty dt_1 \int_0^\infty dt_2 \frac{-2t_1 t_2 + (t_1^2 + t_2^2) \cos(\delta)}{(t_1^2 + 1)(t_2^2 + 1)(t_1^2 + t_2^2 - 2t_1 t_2 \cos(\delta))}, \quad (3.46)$$

where we have performed the change of variable $t_2 \mapsto -t_2$. Next we pose $t_1 = t_2 w$ and we integrate over t_2 . Then we get

$$(c) = -\frac{g^2 N}{4\pi^2} \int_0^\infty dw \frac{\log(w)}{w^2 - 1} \frac{((w^2 + 1) \cos(\delta) - 2w)}{(w^2 + 1 - 2w \cos(\delta))} = -\frac{g^2 N}{4\pi^2} \left(\frac{\pi^2}{4} - \frac{1}{2}(2\pi - \delta)\delta \right). \quad (3.47)$$

Summing the three different contributions, we find the first-order contribution

$$W_1 = (a) + (b) + (c) = \frac{g^2 N}{8\pi^2} (2\pi - \delta)\delta, \quad (3.48)$$

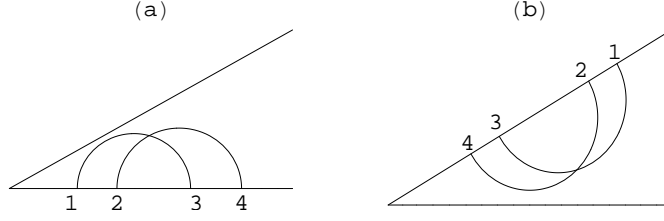


Figure 3.5: Double exchange diagrams: type (I)

consistently with the matrix model prediction (3.17) once one notes that

$$\frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}} = \frac{\delta(2\pi - \delta)}{4\pi^2}. \quad (3.49)$$

The next step is to tackle the double-exchange diagrams, as first contributions at order g^4 . Since the *abelian* part of these diagrams is given by $1/2$ the square of the contribution of order g^2 , as we have seen in the previous section, we shall focus our attention only to the maximally non-abelian part. The relevant diagrams can be separated in three different families, according to the number of propagators with both ends on the same edge of the circuit. To begin with, we have the case of diagrams of fig.(3.5). The contributions of diagram (Ia) and (Ib) are equal. Their value is

$$\begin{aligned} \text{(I)} &= \text{(Ia)} + \text{(Ib)} = 2\text{(Ib)} = \\ &= -\frac{g^4(N^2 - 1)}{8\pi^4} \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \frac{1}{(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)} = \\ &= -\frac{g^4(N^2 - 1)}{3072}. \end{aligned} \quad (3.50)$$

Consider now the second family of diagrams represented in fig.(3.6). Again the two diagrams are equal and we can write

$$\begin{aligned} \text{(II)} &= \text{(IIc)} + \text{(IId)} = 2\text{(IIc)} = \\ &= \frac{g^4(N^2 - 1)}{8\pi^4} \int_0^\infty dt_1 \int_{-\infty}^0 dt_2 \int_{-\infty}^{t_2} dt_3 \int_{-\infty}^{t_3} dt_4 \frac{\cos(\delta)t_1^2 + 2t_3t_1 + \cos(\delta)t_3^2}{(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_1^2 + 2\cos(\delta)t_3t_1 + t_3^2)(t_4^2 + 1)} = \\ &= \frac{g^4(N^2 - 1)}{8\pi^4} \int_0^\infty dt_1 \int_{-\infty}^0 dt_3 \int_{t_3}^0 dt_2 \int_{-\infty}^{t_3} dt_4 \frac{\cos(\delta)t_1^2 + 2t_3t_1 + \cos(\delta)t_3^2}{(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_1^2 + 2\cos(\delta)t_3t_1 + t_3^2)(t_4^2 + 1)}. \end{aligned} \quad (3.51)$$

The integration over t_2 and t_4 are trivial and can be performed analytically. We

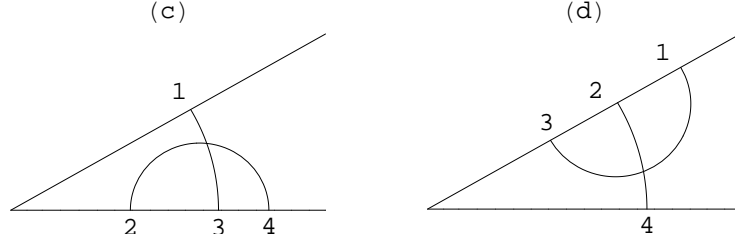
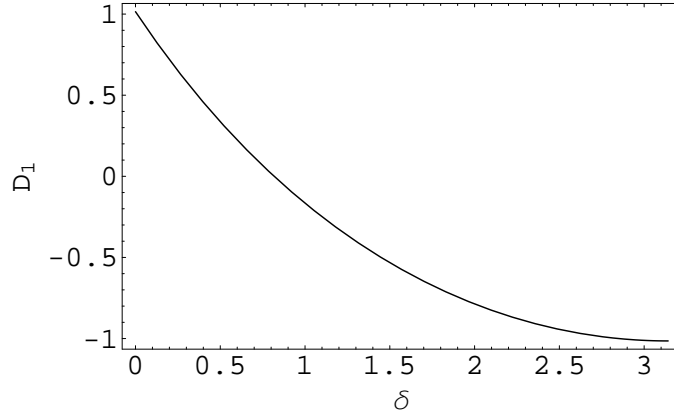


Figure 3.6: Double exchange diagrams: type (II)

Figure 3.7: Plot of \mathcal{D}_1 as a function of δ in the range $[0, \pi]$. We focus our attention to this interval because all the integrals possess the symmetry $\delta \mapsto 2\pi - \delta$.

have

$$\begin{aligned}
 (\text{II}) &= \frac{g^4(N^2 - 1)}{8\pi^4} \int_0^\infty dt_1 \int_{-\infty}^0 dt_3 \frac{\tan^{-1}\left(\frac{1}{t_3}\right) \tan^{-1}(t_3) (\cos(\delta)(t_1^2 + t_3^2) + 2t_3t_1)}{(t_1^2 + 1)(t_3^2 + 1)(t_1^2 + 2\cos(\delta)t_3t_1 + t_3^2)} \equiv \\
 &\equiv \frac{g^4(N^2 - 1)}{8\pi^4} \mathcal{D}_1.
 \end{aligned} \tag{3.52}$$

We could now perform the integration over t_1 since the integrand is a rational function of this variable, but this is not particularly convenient. We would end up indeed with a function of t_1 that we cannot integrate analytically, but only numerically. For this reason, we start our numerical analysis already at the level of \mathcal{D}_1 . The result as a function of δ is given in fig.(3.7).

We consider finally the last diagram contributing to the double-exchange. It is schematically drawn fig.(3.8). The actual integral to evaluate for this diagram is

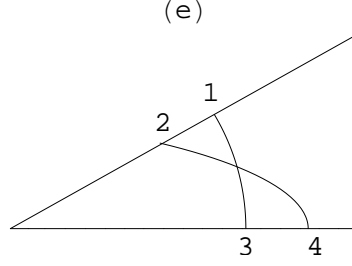
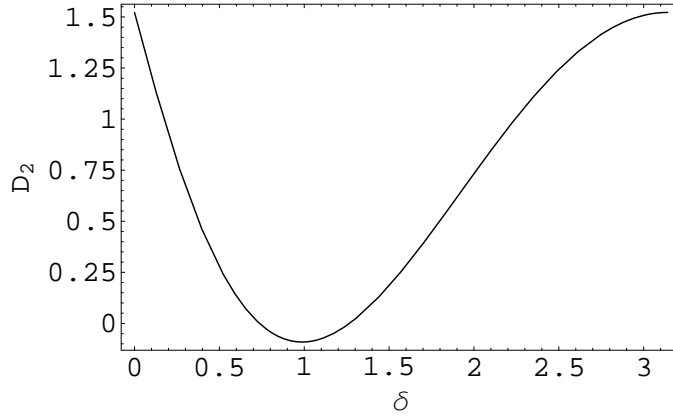


Figure 3.8: Double-exchange diagrams: type (III)

Figure 3.9: Plot of \mathcal{D}_2 as a function of δ in the range $[0, \pi]$.

given by

$$\begin{aligned}
 \text{(III)} &= -\frac{g^4(N^2-1)}{16\pi^4} \int_0^\infty dt_1 \int_0^\infty dt_4 \int_0^{t_1} dt_2 \int_0^{t_4} dt_3 \times \\
 &\times \frac{((t_1^2+t_3^2)\cos(\delta)-2t_1t_3)((t_2^2+t_4^2)\cos(\delta)-2t_2t_4)}{(t_1^2+1)(t_2^2+1)(t_3^2+1)(t_4^2+1)(t_1^2-2t_3\cos(\delta)t_1+t_3^2)(t_2^2-2t_4\cos(\delta)t_2+t_4^2)} \equiv -\frac{g^4(N^2-1)}{16\pi^4} \mathcal{D}_2,
 \end{aligned}
 \tag{3.53}$$

where we have performed the following change of variables $t_3 \mapsto -t_3$ and $t_4 \mapsto -t_4$ and then we have rearranged the order of the different integrations. In this form, the integration over t_2 and t_3 can be performed analytically, while the residual two integrations can be done numerically. The final result for \mathcal{D}_2 is plotted in fig.(3.9).

Having evaluated the double-exchange diagrams, we are left to consider the effective contribution due to the interactions and summarized in the result (3.39) found in the previous subsection. In order to write the actual integrals we have to compute, we distinguish two cases: (A) when the legs of the vertex are attached to the same edge of the circuit (see fig.(3.10)) and (B) when the legs of the vertex are not attached to the same edge of the circuit (see figs.(3.11),(3.12)). The diagrams

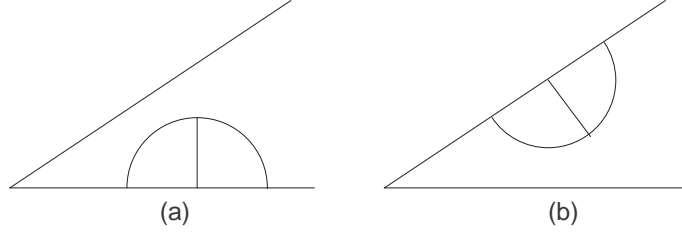


Figure 3.10: Diagrams of type (A)

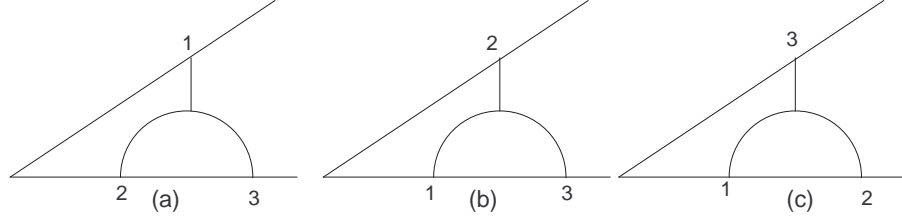


Figure 3.11: Diagrams of type (B): set(1)

belonging to the family (A) vanish. To convince the reader let us consider , for example, the first of the two diagrams that it is given by

$$\frac{g^4(N^2 - 1)}{64\pi^4} \int_{-\infty}^0 dt_1 \int_{-\infty}^0 dt_2 \int_{-\infty}^0 dt_3 \epsilon(t_1, t_2, t_3) \frac{(t_2 t_3 + 1) \log \left(\frac{(t_1 - t_2)^2 (t_3^2 + 1)}{(t_2^2 + 1)(t_1 - t_3)^2} \right)}{(t_1^2 + 1)(t_2^2 + 1)(t_2 - t_3)(t_3^2 + 1)}. \quad (3.54)$$

This integral is equal to zero because the integrand is antisymmetric in the interchange (t_2, t_3) , while the integration region is symmetric. We are finally left with the diagrams belonging to the family (B). We have six diagrams: (1) three with two legs of the vertex attached to first edge of the spherical wedge and (2) three with two legs attached to second edge. The contribution of the two classes is equal because our loop is symmetric under reflection with respect the longitude $\phi = \delta/2$. We shall consider the first class only and we will multiply the result by two. Then the total contribution of the interaction is given by the following integral

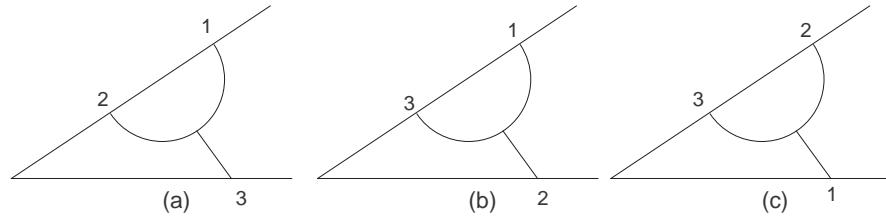


Figure 3.12: Diagrams of type (B): set (2)

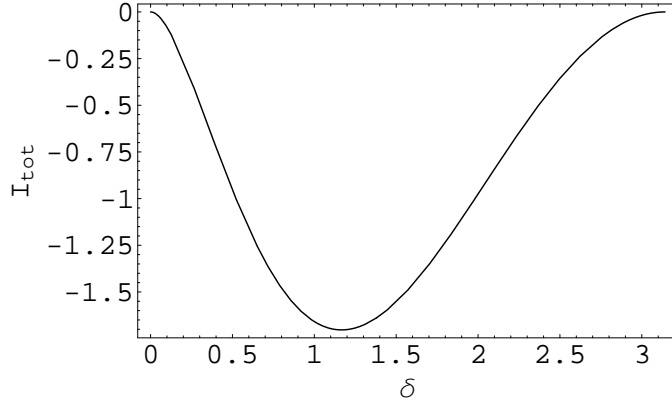


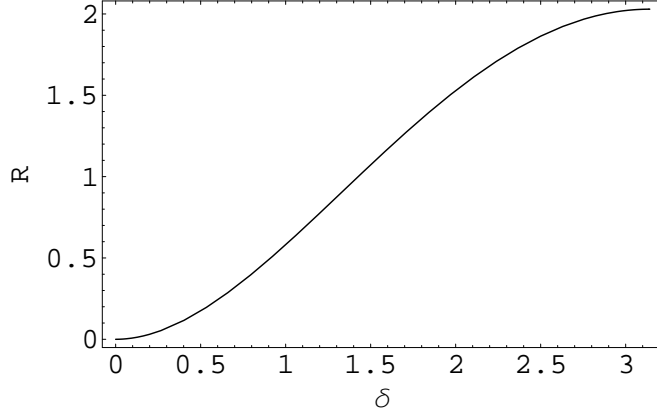
Figure 3.13: Plot of I_{tot} as a function of δ in the range $[0, \pi]$.

$$\begin{aligned}
 \mathcal{I}_{tot} &= \frac{g^4(N^2 - 1)}{64\pi^4} \int_0^\infty dt_1 dt_2 dt_3 [(\text{sgn}(t_3 - t_2)V_1 + \text{sgn}(t_1 - t_3)V_2 + \text{sgn}(t_2 - t_1)V_3] \equiv \\
 &\equiv \frac{g^4(N^2 - 1)}{64\pi^4} I_{tot},
 \end{aligned} \tag{3.55}$$

where

$$\begin{aligned}
 V_1 &= \frac{2(t_2 t_3 + 1)((t_1^2 + t_3^2) \cos(\delta) - 2t_1 t_3) \log\left(\frac{(t_3^2 + 1)(t_1^2 - 2t_2 \cos(\delta)t_1 + t_2^2)}{(t_2^2 + 1)(t_1^2 - 2t_3 \cos(\delta)t_1 + t_3^2)}\right)}{(t_1^2 + 1)(t_2^2 + 1)(t_3 - t_2)(t_3^2 + 1)(t_1^2 - 2t_3 \cos(\delta)t_1 + t_3^2)} \\
 V_2 &= \frac{2(t_2(t_3^2 - 1) - (t_2^2 - 1)t_3 \cos(\delta)) \log\left(\frac{(t_3^2 + 1)(t_1^2 - 2t_2 \cos(\delta)t_1 + t_2^2)}{(t_2^2 + 1)(t_3 - t_1)^2}\right)}{(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_2^2 - 2t_3 \cos(\delta)t_2 + t_3^2)} \\
 V_3 &= \frac{2((t_2^2 - 1)t_3 \cos(\delta) - t_2(t_3^2 - 1))((t_1^2 + t_3^2) \cos(\delta) - 2t_1 t_3) \log\left(\frac{(t_2 - t_1)^2(t_3^2 + 1)}{(t_2^2 + 1)(t_1^2 - 2t_3 \cos(\delta)t_1 + t_3^2)}\right)}{(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_1^2 - 2t_3 \cos(\delta)t_1 + t_3^2)(t_2^2 - 2t_3 \cos(\delta)t_2 + t_3^2)}.
 \end{aligned} \tag{3.56}$$

If we expand the logarithms in this expression, we can always perform one integration analytically: the argument of each logarithm always depends only on two of the three variables. It turns out that one of the three integrations always reduces to find the primitive of a rational function. As we have done before, the remaining two integrations can be easily performed numerically and the result is given in fig.(4.9). We can finally collect all the results to obtain the maximal nonabelian

Figure 3.14: Plot of R as a function of δ in the range $[0, \pi]$.

contribution at order g^4 :

$$\begin{aligned}
 W_2^{mnb} &= -\frac{g^4(N^2-1)}{3072} + \frac{g^4(N^2-1)}{8\pi^4}\mathcal{D}_1 - \frac{g^4(N^2-1)}{16\pi^4}\mathcal{D}_2 + \frac{g^4(N^2-1)}{64\pi^4}\mathcal{I}_{tot} = \\
 &= -\frac{g^4(N^2-1)}{8\pi^4} \left(\frac{\pi^4}{384} - \mathcal{D}_1 + \frac{1}{2}\mathcal{D}_2 - \frac{1}{8}\mathcal{I}_{tot} \right) \equiv -\frac{g^4(N^2-1)}{8\pi^4}R.
 \end{aligned} \tag{3.57}$$

The plot for R is given in fig. 5.7. We can easily perform a fit of the numerical result R with a polynomial of the following form $P(\delta) = c_0\delta^2(2\pi - \delta)^2$. This particular dependence is necessary in order to be in agreement with the conjectured relation with the zero-instanton sector of pure Yang-Mills theory on the sphere. The coefficient c_0 is easily determined and it is $1/(48)$. The difference between R and the polynomial P is less than 10^{-8} over the whole range of the value of δ .

Thus we have

$$W_2^{mnb} = -\frac{g^4(N^2-1)}{8 \cdot 48\pi^4}\delta^2(2\pi - \delta)^2 = -\frac{g^4(N^2-1)}{24} \frac{\mathcal{A}_1^2 \mathcal{A}_2^2}{\mathcal{A}^4} \tag{3.58}$$

and after the inclusion of the abelian contribution

$$\langle W \rangle_{g^4} = \frac{g^4 N^2}{8} \frac{\mathcal{A}_1^2 \mathcal{A}_2^2}{\mathcal{A}^4} - \frac{g^4(N^2-1)}{24} \frac{\mathcal{A}_1^2 \mathcal{A}_2^2}{\mathcal{A}^4} = \frac{g^4(2N^2+1)}{24} \frac{\mathcal{A}_1^2 \mathcal{A}_2^2}{\mathcal{A}^4} \tag{3.59}$$

coincides exactly with the g^4 order expansion of the YM_2 prediction (3.18) [71].

3.4 Wilson loops at Strong Coupling

In this subsection we review the construction of the string solutions dual to this class of Wilson loops defined on S^2 . We begin with the simple case of the 1/4 BPS

latitude well studied in [84, 85] and then we will analyze the operator duals to the "wedge" Wilson loops. Afterwards we will evaluate the string action on these solutions to confirm at strong coupling the conjecture proposed in the previous subsections.

Since all the operators that belong to this class are restricted on a S^3 inside S^4 ($x_4 = 0$) the dual solutions will lie on a subspace of $AdS_5 \times S^5$, namely on $AdS_4 \times S^2$. Its metric can be written as

$$ds^2 = \frac{L^2}{y^2} (dy^2 + dr^2 + r^2 d\phi^2 + dx_3^2) + L^2 (d\delta^2 + \sin^2 \delta d\varphi^2) \quad (3.60)$$

where the first part is the metric of the AdS_4 space with (r, ϕ) the polar coordinates in the plane (x_1, x_2) and the second one is the standard metric on S^2 . The boundary of the string ends along a closed loop C described by

$$x_\mu = (0, r = \sin \theta_0, \phi = \tau, x_3 = \cos \theta_0) \quad 0 < \tau < 2\pi \quad (3.61)$$

and $\delta_0 = \pi/2 - \theta_0$. To calculate the action we make the ansatz

$$y = y(\sigma) \quad r = r(\sigma) \quad \delta = \delta(\sigma) \quad \phi = \varphi + \pi = \tau \quad (3.62)$$

where $0 < \sigma < \infty$ and τ are the coordinates of the worldsheet and where the phase difference of π between ϕ and φ is a consequence of the supersymmetric construction on the gauge side. The Lagrangian in conformal gauge reads

$$\mathcal{L} = \frac{L^2}{y^2} \left[\left(\frac{dy}{d\sigma} \right)^2 + \left(\frac{dr}{d\sigma} \right)^2 + r^2 \right] + L^2 \left[\left(\frac{d\delta}{d\sigma} \right)^2 + \sin^2(\delta) \right]. \quad (3.63)$$

The solutions to the equations of motion (3.65, 3.66, 3.67) and to Virasoro constraint (3.64)

$$\left(\frac{dy}{d\sigma} \right)^2 + \left(\frac{dr}{d\sigma} \right)^2 + \left(\frac{d\delta}{d\sigma} \right)^2 y^2 - r^2 - y^2 \sin^2 \delta = 0 \quad (3.64)$$

$$\left(\frac{d^2 y}{d\sigma^2} \right) y - \left(\frac{dy}{d\sigma} \right)^2 + \left(\frac{dr}{d\sigma} \right)^2 + r^2 = 0 \quad (3.65)$$

$$\left(\frac{d^2 r}{d\sigma^2} \right) - \frac{2}{y} \left(\frac{dy}{d\sigma} \right) \left(\frac{dr}{d\sigma} \right) - r = 0 \quad (3.66)$$

$$\left(\frac{d^2 \delta}{d\sigma^2} \right) - \sin \delta \cos \delta = 0 \quad (3.67)$$

are given by

$$y = \sin \theta_0 \tanh \sigma \quad r = \frac{\sin \theta_0}{\cosh \sigma} \quad \sin \delta = \frac{1}{\cosh(\sigma_0 + \sigma)} \quad (3.68)$$

where the integration constant σ_0 is fixed by demanding that at $\sigma = 0$ we have $\sin \delta_0 = 1/\cosh \sigma_0 = \cos \theta_0$. Evaluating the classical action on (3.68), after the addition of the boundary term, we obtain

$$S = \sqrt{\lambda} \int_0^\infty d\sigma \left(\frac{1}{\cosh^2 \sigma \sinh^2 \sigma} - \frac{1}{\sinh^2 \sigma} + \frac{1}{\cosh^2 (\sigma_0 + \sigma)} \right) = -\sin \theta_0 \sqrt{\lambda} \quad (3.69)$$

and thus the *VEV* of the Wilson Loop at strong coupling is equal to

$$\langle W \rangle \simeq \exp \left[\sqrt{\lambda} \sin \theta_0 \right] \quad (3.70)$$

that matches with the prediction (3.17). To be more precise there is another supersymmetric configuration that corresponds to a worldsheet that wraps the other part of the dual $S^2 \subset S^5$ (and with opposite regularized action respect to (3.69)):

$$y = \sin \theta_0 \tanh \sigma \quad r = \frac{\sin \theta_0}{\cosh \sigma} \quad \sin \delta = \frac{1}{\cosh (\sigma_0 - \sigma)} \quad (3.71)$$

Both solutions, the stable (3.68) and the unstable one (3.71), appear as saddle points in the larger λ limit of the gaussian matrix model.

Now let us consider the configuration dual to a quarter BPS Wilson loop made of two longitudes at angle δ . The basic idea for finding this minimal surface is the following; one solves the equations of motion for a configuration made by a cusp in the origin with opening angle δ (see fig.(3.3)) and afterwards performs a conformal transformation that map the cusp to the wedge on S^2 .

To begin the analysis we consider the $AdS_3 \times S^1$ subspace with a metric given by

$$ds^2 = \frac{L^2}{y^2} (dy^2 + dr^2 + r^2 d\phi^2) + L^2 d\varphi^2 \quad (3.72)$$

Then in terms of the worldsheet coordinates σ and τ we make the following ansatz

$$\begin{aligned} y &= \rho(\tau) \sin(\nu(\sigma)) & \phi &= \phi(\sigma) \\ r &= \rho(\tau) \cos(\nu(\sigma)) & \varphi &= \varphi(\sigma). \end{aligned} \quad (3.73)$$

The lagrangian in conformal gauge thus reads

$$\mathcal{L} = \frac{1}{\rho^2 \sin^2 \nu} \left(\frac{d\rho}{d\tau} \right)^2 + \frac{1}{\sin^2 \nu} \left(\frac{d\nu}{d\sigma} \right)^2 + \frac{\cos^2 \nu}{\sin^2 \nu} \left(\frac{d\phi}{d\sigma} \right)^2 + \left(\frac{d\varphi}{d\sigma} \right)^2 \quad (3.74)$$

and the Virasoro constraint is given by

$$\frac{1}{\rho^2} \left(\frac{d\rho}{d\tau} \right)^2 = \left(\frac{d\nu}{d\sigma} \right)^2 + \cos^2(\nu) \left(\frac{d\phi}{d\sigma} \right)^2 + \sin^2(\nu) \left(\frac{d\varphi}{d\sigma} \right)^2. \quad (3.75)$$

From the equation of motion for ρ , it is straightforward to obtain $\rho = s_0 e^{ap\tau}$. Subsequently we can recast the Virasoro constraint (3.75) in the more manageable form

$$a^2 p^2 = \left(\frac{d\nu}{d\sigma} \right)^2 + \cos^2 \nu \left(\frac{d\phi}{d\sigma} \right)^2 + \sin^2 \nu \left(\frac{d\varphi}{d\sigma} \right)^2. \quad (3.76)$$

Since in (3.74) the ϕ and φ dependence is trivial we can find two conserved quantities (the canonical momenta conjugates to them)

$$\mathcal{P}_\phi = \frac{\cos^2(\nu)}{\sin^2(\nu)} \left(\frac{d\phi}{d\sigma} \right) \quad \mathcal{P}_\varphi = \left(\frac{d\varphi}{d\sigma} \right). \quad (3.77)$$

In the BPS case one can show that $\mathcal{P}_\phi = \mathcal{P}_\varphi$ and the system of equations becomes more simply to treat. Indeed, we can set $a = \mathcal{P}_\phi = \mathcal{P}_\varphi$ and as a consequence we find that (3.75) takes the form:

$$\left(\frac{d\nu}{d\sigma} \right)^2 = a^2 (p^2 - \tan^2 \nu), \quad (3.78)$$

that can be easily integrated and gives

$$\sin \nu = \frac{p \sin \left(a \sqrt{p^2 + 1} \sigma \right)}{\sqrt{p^2 + 1}}. \quad (3.79)$$

Now it is simple to solve the other equations and we find

$$\begin{aligned} \left(\frac{d\varphi}{d\sigma} \right) = a &\longrightarrow \varphi = \sigma a \\ \frac{\cos^2 \nu}{\sin^2 \nu} \left(\frac{d\phi}{d\sigma} \right) = a &\longrightarrow \tan(\phi + a\sigma + c_1) = \frac{\tan(a\sigma\sqrt{1+p^2})}{\sqrt{1+p^2}}. \end{aligned}$$

The constant p is related to the angle of the cusp δ in the following way. The variable ϕ is the polar angle on the plane defined by the cusp and thus on the boundary ($y = 0$) we have that for $\sigma = 0$, $\phi = 0$ while for $\sigma = \pi$ we must impose $\phi = \delta$. We obtain $c_1 = 0$, $a = \frac{1}{\sqrt{1+p^2}}$ and

$$\delta = \pi \left(1 - \frac{1}{\sqrt{1+p^2}} \right) \quad (3.80)$$

At this point we have to perform a conformal transformation that maps our solution on the two sphere. In other words since a conformal transformation in the gauge

theory is reflected into an isometry in AdS_5 we have to find an isometry on this space that reduces to the usual stereographic projection on the boundary. Working in global AdS coordinate R, θ, ϕ, φ , where the metric is

$$ds^2 = L^2(dR^2 + \sinh^2 R(d\theta^2 + \sin^2 \theta d\phi^2) + d\varphi^2) \quad (3.81)$$

the conformal transformation send θ and ϕ into itself and

$$\cosh R = \frac{(1 + \rho^2)}{2\rho \sin \nu} \quad \sin \theta \sinh R = \cot \nu \quad (3.82)$$

After the inclusion of the Jacobian, the action in the new coordinates reads

$$\mathcal{S} = \frac{\sqrt{\lambda}}{2\pi} \int dR d\theta \frac{p \sinh^2 R \sin \theta}{\sqrt{p^2 \sinh^2 R \sin^2 \theta - 1}}. \quad (3.83)$$

This terms is naturally divergent and to make it finite we have to add as usual a boundary term, namely the Legendre transform of the original Lagrangian with respect to the coordinates orthogonal to the boundary. The boundary terms that we have to add is

$$\mathcal{S}_B = -\frac{\sqrt{\lambda}}{2\pi} \int d\theta \frac{p^2 \sin \theta \sinh R^2(1 + \sinh^2 R \sin^2 \theta) - \sin \theta \cosh R^2}{p(\sinh^2 R \sin^2 \theta + 1)\sqrt{p^2 \sinh^2 R \sin^2 \theta - 1}} \quad (3.84)$$

that together with (3.83) give

$$\begin{aligned} \mathcal{S} &= \sqrt{\lambda} \left(\cosh R - \sinh R - \sqrt{1 + \frac{1}{p^2}} + \frac{\coth R}{p\sqrt{1 + p^2}} \right)_{R \rightarrow \infty} \\ &= -\frac{\sqrt{\lambda}}{\pi p} \left(\sqrt{1 + p^2} - \frac{1}{\sqrt{1 + p^2}} \right). \end{aligned} \quad (3.85)$$

Using eq. (3.80) we immediately obtain

$$S = -\frac{\sqrt{\lambda\delta(2\pi - \delta)}}{\pi}. \quad (3.86)$$

So the VEV of a Wilson loop on a circuits made by two longitude at angle δ is given by

$$\langle W \rangle \simeq \exp \left[\frac{\sqrt{\lambda\delta(2\pi - \delta)}}{\pi} \right] \quad (3.87)$$

that is the same behavior of (3.17). Again to match the numerical coefficient we have to take into account the zero modes of the fluctuations around this minimal surface.

Despite these particular examples, a systematic study of the solution dual to the operators defined on S^3 has been done in [60, 80]. A nice geometrical picture comes out in the analysis of [60] : the authors have shown that the dual string worldsheets are pseudoholomorphic surfaces with respect to an almost complex structure J defined on $AdS_4 \times S^2$. Writing the metric of this subspace as

$$ds^2 = \frac{1}{z^2} dx^\mu dx^\mu + z^2 dy^i dy^i \quad (3.88)$$

with $\mu = 1..4$, $i = 1, 2, 3$ and where $z^{-2} = y^i y^i$, the string solution $X^M(\tau, \sigma)$ satisfies the equation

$$J_N^M \partial_\alpha X^N = \sqrt{g} \epsilon_{\alpha\beta} \partial^\beta X^M \quad (3.89)$$

where $M = 1..7$, \sqrt{g} is the determinant of the worldsheet metric, ϵ is the usual antisymmetric symbol ($\epsilon_{\tau\sigma} = -1$) and the almost complex structure J in component is given by [60]

$$J_\mu^\nu = z^2 \sigma_{\mu\nu}^i y^i \quad J_i^\mu = z^2 \sigma_{\mu\nu}^i x^\nu = -z^4 J_\mu^i \quad J_j^i = -z^2 \epsilon_{ijk} y^k. \quad (3.90)$$

Intriguingly the author of [60] found that the solutions of (3.89) are supersymmetric and automatically satisfy the Virasoro constraints and the equations of motion for the $AdS_5 \times S^5$ σ -model. Moreover they have explicitly shown that these configurations preserve the same supercharges of the dual Wilson loops on S^3 . The action of a surface obeying (3.89) is given by

$$\mathcal{S} = \frac{\sqrt{\lambda}}{2\pi} \int_\Sigma \mathcal{J} \quad (3.91)$$

where \mathcal{J} is the two-form associated with the complex structure J . To be more precise from (3.91) we must subtract a boundary term that cancels out the divergences and as usual reads

$$\mathcal{S}_B = \int_{\partial\Sigma} d\tau z \frac{d\mathcal{L}}{dz} \quad (3.92)$$

with z the radial AdS coordinate (orthogonal to the boundary).

Another interesting check of the conjecture at strong coupling has been found in [80]. Indeed it was shown that under an arbitrary small perturbation of the boundary loop $C \rightarrow C + \delta C$ such that both the loop and the perturbed one live on a S^2 , the variation of the action reads

$$\delta S = -F\delta A \tag{3.93}$$

where F is a constant and where A is the area enclosed by the loop C . In other terms the authors prove that the string action is invariant under an arbitrary deformation of C that leaves the area unchanged. This fact is very nice since the conjecture $\mathcal{N}=4$ SYM/ YM_2 is expected to hold at strong coupling and thus the string configurations must have the same properties of the YM_2 theory (that, as we know, is invariant under area preserving diffeomorphism).

Chapter 4

Wilson Loops on Hyperbolic Space

4.1 Introduction

In this chapter we will discuss a family of Wilson loops in Minkowski space-time that has been considered for the first time in [90] and that is closely related to the one constructed on S^3 . The loops lie on a three dimensional hyperbolic space \mathbb{H}_3 (Euclidean AdS_3) and generically they are 1/16 BPS operators. In particular we will focus on a sub-class of these operators, namely when the loops are restricted to live on $\mathbb{H}_2 \subset \mathbb{H}_3$. Such loops are interesting since they are asymptotic to the light-cone and thus they are similar to light-like cusped Wilson loops that have been used to calculate scattering amplitudes. Moreover, as we will see they are expected to be fully captured by a gaussian matrix model.

Indeed a first order computation suggests (at least for compact shape) the equivalence between the $\mathcal{N} = 4$ SYM calculation and the analogous computation in YM_2 on a \mathbb{H}_2 , captured by [90, 92]

$$\langle W \rangle_{YM_2} = \frac{1}{N} L_{N-1}^1 \left(g_{2d}^2 \frac{\mathcal{A}(\mathcal{A} + 4\pi)}{4\pi} \right) \exp \left[-\frac{g_{2d}^2}{2} \frac{\mathcal{A}(\mathcal{A} + 4\pi)}{8\pi} \right], \quad (4.1)$$

after the identification of the couplings

$$g_{2d}^2 = \frac{g^2}{4\pi} \quad (4.2)$$

and where \mathcal{A} is the area of the region enclosed by the loop. For non compact loops the situation is more involved since it is not clear how and if the reduction to the 2d Yang Mills works.

In the following sections, after the analysis of the supersymmetric properties of these observables, we perform a perturbative two-loop computation in order to check the relation with the result (4.1)(at least perturbatively).

4.2 Supersymmetric Wilson loops on \mathbb{H}_3

The Wilson loops that we will analyze are defined on a hyperbolic sub-manifold of the Minkowski space [90, 91] defined by the constraint

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1. \quad (4.3)$$

Essentially these operators are constructed from those that live on a three sphere presented in chapter three by a Wick-rotation; indeed the invariant one-form defined on a two sphere (3.2a), (3.2b) and (3.2c) can be rotated into

$$\omega_1 = x^0 dx^1 - x^1 dx^0 + i(x^2 dx^3 - x^3 dx^2) \quad (4.4)$$

$$\omega_2 = x^0 dx^2 - x^2 dx^0 + i(x^3 dx^1 - x^1 dx^3) \quad (4.5)$$

$$\omega_3 = x^0 dx^3 - x^3 dx^0 + i(x^1 dx^2 - x^2 dx^1) \quad (4.6)$$

to define the couplings to the three scalars. The Wilson loop in terms of the modified connection

$$\tilde{A} = A_\mu dx^\mu + i \omega_i M_I^i \phi^I \quad (4.7)$$

where M_I^i is a 3×6 matrix (with MM^\top is the 3×3 unit matrix), can be written as

$$W = \frac{1}{N} \text{Tr} \mathcal{P} \exp \oint i \tilde{A} ds. \quad (4.8)$$

If $\epsilon(x) = \epsilon_0 + x^\mu \gamma_\mu \epsilon_1$ is the conformal Killing spinor with ϵ_0 and ϵ_1 constant spinors that generate respectively the supersymmetric and the superconformal transformations, the variation of a generic loop on \mathbb{H}_3 can be recast in this manner

$$\begin{aligned} i\tau^i \epsilon_1^+ &= M_I^i \rho^I \epsilon_0^+ \\ \epsilon_1^- &= \epsilon_0^- = 0 \end{aligned} \quad (4.9)$$

where τ^i are the Pauli matrices, the ρ^i belong to the Clifford Algebra of $SO(6)$ and $\epsilon_{0,1}^\pm = \frac{1}{2}(1 \pm \gamma_5)\epsilon_{0,1}$. In the first line one can read three independent equations that together with the constraints $\epsilon_1^- = \epsilon_0^- = 0$ reduce to two the number of supercharges preserved. Restricting the curve to \mathbb{H}_2 doubles the number of supersymmetries;

indeed in that case, the vanishing of the supersymmetric variation of the operator gives

$$\begin{aligned} i\gamma_{01} \epsilon_1 &= \rho^1 \gamma^5 \epsilon_0 \\ i\gamma_{02} \epsilon_1 &= \rho^2 \gamma^5 \epsilon_0 \\ \gamma_{12} \epsilon_1 &= \rho^3 \gamma^5 \epsilon_0, \end{aligned} \tag{4.10}$$

and thus a generic contour on this subspace will be 1/8 BPS object. In the same notation of *appendix A*, the preserved supercharges read

$$Q^a = (i\tau_2)_\alpha^\alpha Q_{\alpha}^{\dot{a}a} + (\tau_2)_\alpha^{\dot{a}} \epsilon^{ab} S_{ab}^\alpha \quad \bar{Q}^a = (-i\tau_1)_{\dot{\alpha}\dot{a}} Q^{\dot{a}a} + (\tau_1)^{\dot{a}\dot{a}} \epsilon^{ab} S_{\dot{a}\dot{b}}^a \tag{4.11}$$

($a = 1, 2$). If instead we choose a particular contour, supercharges preserved can be more than four, for example the hyperbolic line and the circle are respectively 1/2 and 1/4 BPS objects [90].

The two-loop computation for a generic circuit appears cumbersome to analyze and so we choose a particular shape for the Wilson loop. In the next section we study numerically an operator defined on a circuit \mathcal{C} made by two finite rays with a cusp in the origin at opening angle δ plus an arc that closes the contour (see fig. 4.1). This circuit is the "analogous" of a truncated wedge on the two sphere. We parameterize the contour \mathcal{C} as follows

$$x^\mu = (\cosh t, \sinh t, 0, 0) \quad -\theta < t < 0 \tag{4.12a}$$

$$y^\mu = (\cosh t, -\sinh t \cos \delta, -\sinh t \sin \delta, 0) \quad 0 < t < \theta \tag{4.12b}$$

$$z^\mu = (\cosh \theta, \sinh \theta \cos t, \sinh \theta \sin t, 0) \quad \pi - \delta < t < \pi \tag{4.12c}$$

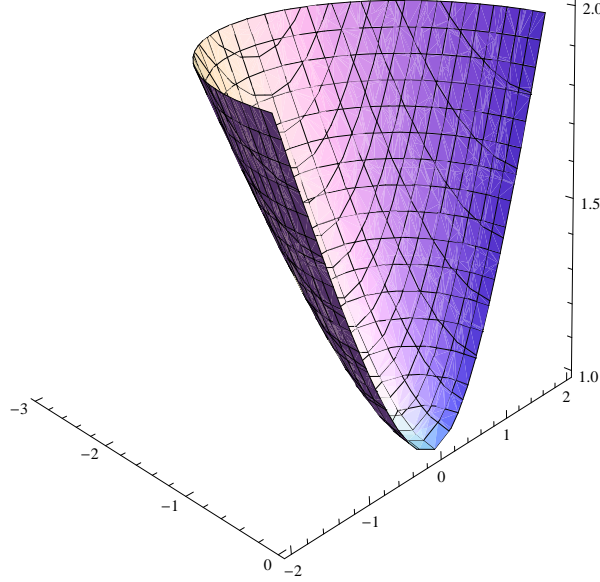
where θ is the hyperbolic angle of the arc and δ is the opening angle of the cusp. Along the three edges the couplings with the scalars are defined trough the one-forms

$$\omega_x = (1, 0, 0) \tag{4.13}$$

$$\omega_y = (-\cos \delta, \sin \delta, 0) \tag{4.14}$$

$$\omega_z = (-\cosh \theta \sinh \theta \sin t, \cosh \theta \sinh \theta \cos t, i \sinh^2 \theta) \tag{4.15}$$

Since the supersymmetric variation for this particular type of Wilson loop can be rearranged as in (4.10) (case of the generic circuit), these operators are 1/8 BPS objects.

Figure 4.1: *Area* on \mathbb{H}_2 delimited by the Wilson Loop

4.3 Perturbative Calculation

Two-loop expansion for Wilson loops on \mathbb{H}_2

In this section we discuss the expansion at the first two perturbative orders of a supersymmetric Wilson loops lying on the contour defined in (4.12) inside \mathbb{H}_2 . We work in Feynman gauge and we will need to adopt a regularization procedure (dimensional regularization) since, as we will see, divergent diagrams appear in intermediate steps of the computations. Nonetheless the final results for a compact contour will be finite like it happens for the case of the Wilson loop on the two-sphere [75].

We begin considering the effective gluon-scalar propagator that in 2ω dimensions reads as

$$\Delta_{ij}^{ab}(\tau_1, \tau_2) = \frac{\Gamma(w-1)}{4\pi^w} \delta^{ab} \frac{-x_i(\tau_1) \cdot x_j(\tau_2) + w_i(\tau_1) \cdot w_j(\tau_2)}{((x_i(\tau_1) - x_j(\tau_2))^2)^{w-1}}. \quad (4.16)$$

Since at g^2 order the equivalence with the YM_2 computation is shown in reference [90] here we directly pass to analyze the g^4 order where interactions must be taken into account. Different terms contribute to the expectation value and not all of them are finite; for this reason a dimensional regularization procedure has to be introduced. We skip technicalities of the divergencies cancelation and report them

in the *appendix F* since they are quite similar to the case of the Wilson loops on the two sphere well analyzed in the previous chapter.

In summary when one considers the term coming from the so called bubble diagrams, the divergent part cancels exactly the divergence of the three vertex (of the type AAA + $A\phi\phi$) and finds a completely finite result. The only surviving contribution \mathcal{I}_{tot} from the sum of the spider and the bubble can be written as a relatively simple convergent integral as

$$\mathcal{I}_{\text{tot}} = \frac{g^4(N^2 - 1)}{128\pi^4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) \frac{(\omega_{x_3} \cdot \omega_{x_1} - \dot{x}_1 \cdot \dot{x}_3)}{(x_3 - x_1)^2} \times \frac{(x_3 - x_2) \cdot \dot{x}_2}{(x_3 - x_2)^2} \log \left(\frac{(x_2 - x_1)^2}{(x_3 - x_1)^2} \right). \quad (4.17)$$

Then we have to consider the *double-exchange* diagrams in the perturbative expansion of the Wilson loop that can be written as

$$\frac{g^4}{N} \oint_C dt_1 dt_2 dt_3 dt_4 \theta(t_1 - t_2) \theta(t_2 - t_3) \theta(t_3 - t_4) \langle \text{Tr}[\mathcal{A}(t_1) \mathcal{A}(t_2) \mathcal{A}(t_3) \mathcal{A}(t_4)] \rangle_0. \quad (4.18)$$

The effective propagator has the color structure $\Delta^{ab}(t_1, t_2) = \delta^{ab} \Delta(t_1, t_2)$ and the relevant Green function can be written as

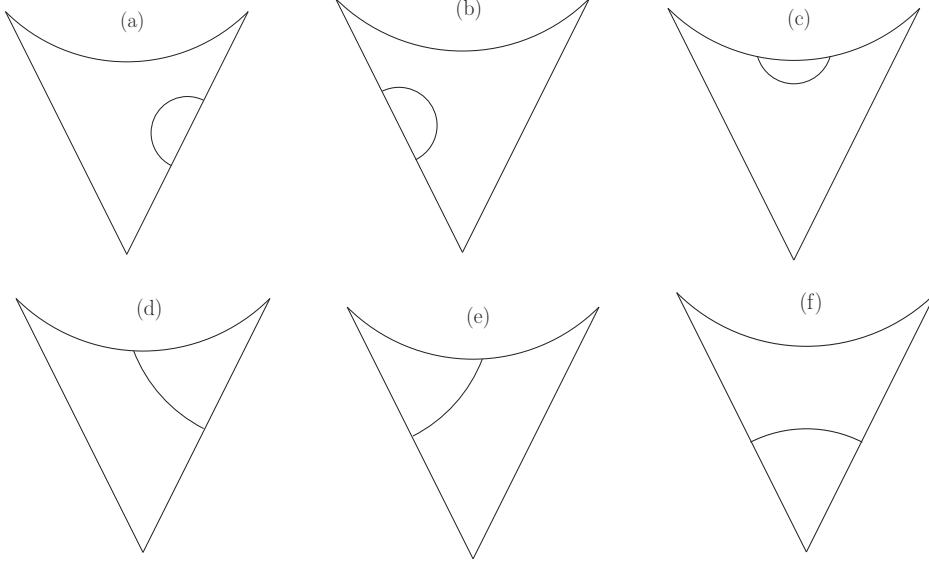
$$\begin{aligned} \langle \text{Tr}[\mathcal{A}(t_1) \mathcal{A}(t_2) \mathcal{A}(t_3) \mathcal{A}(t_4)] \rangle_0 &= \frac{1}{2} \text{Tr}([T^b, T^a][T^b, T^a]) \Delta(t_1, t_3) \Delta(t_2, t_4) + \\ &+ \text{Tr}(T^a T^a T^b T^b) [\Delta(t_1, t_2) \Delta(t_3, t_4) + \Delta(t_1, t_3) \Delta(t_2, t_4) + \Delta(t_1, t_4) \Delta(t_2, t_3)]. \end{aligned} \quad (4.19)$$

As usual this formula can be rearranged as a sum of two contributions: the so-called *abelian part* that is exactly 1/2 the square of the single-exchange contribution and the *maximally non-abelian part* respectively given by

$$\frac{1}{2} \left(\frac{g^2 N}{8\pi^2} \oint_C dt_1 dt_2 \frac{(\omega_{x_1} \cdot \omega_{x_2} - \dot{x}_1 \cdot \dot{x}_2)}{(x_1 - x_2)^2} \right)^2 \quad (4.20)$$

and

$$- \frac{g^4(N^2 - 1)}{64\pi^4} \oint_C dt_1 dt_2 dt_3 dt_4 \theta(t_1 - t_2) \theta(t_2 - t_3) \theta(t_3 - t_4) \times \frac{(\omega_{x_1} \cdot \omega_{x_3} - \dot{x}_1 \cdot \dot{x}_3)(\omega_{x_2} \cdot \omega_{x_4} - \dot{x}_2 \cdot \dot{x}_4)}{(x_1 - x_3)^2 (x_4 - x_2)^2}. \quad (4.21)$$

Figure 4.2: *Single-exchange* diagrams

Numerical Calculation

In the present subsection we will provide a fourth-order numerical evidence that the supersymmetric Wilson loops lying on \mathbb{H}_2 are equivalent to the usual, non-supersymmetric Wilson loops in the zero instanton sector of the YM_2 on \mathbb{H}_2 . We consider the loop as described in (4.12) with a fixed θ angle (we analyze three different $\theta = 0.75, 1, 1.25$). Let us start by discussing the lowest order contribution. At this level the diagrams that contribute are those in figure (4.2). The diagrams (a) and (b) are equal, since we cannot distinguish the two rays. We have that

$$(a) + (b) = 2(a) = -\frac{g^2 N}{16\pi^2} \int_0^\theta dt_1 dt_2 = -\frac{\lambda \theta^2}{16\pi^2}. \quad (4.22)$$

The diagram (c) is given by

$$(c) = -\frac{g^2 N \sinh \theta}{32\pi^2} \int_0^\delta dt_1 dt_2 = -\frac{\lambda \sinh \theta \delta^2}{32\pi^2} \quad (4.23)$$

Diagrams (d) and (e) are also equal

$$(d) = (e) = \int_0^\theta \int_{\pi-\delta}^\pi \frac{\lambda \sinh(\theta) (\cosh(t_1) - \cosh(\theta)) \sin(t_2 + \delta)}{16\pi^2 (\sinh(t_1) \sinh(\theta) \cos(t_2 + \delta) + \cosh(t_1) \cosh(\theta) - 1)} dt_1 dt_2 \quad (4.24)$$

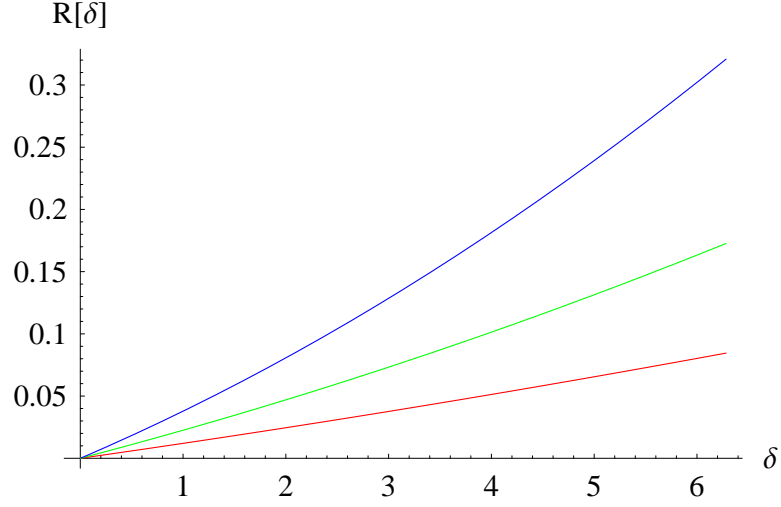


Figure 4.3: Plot of $R[\delta]$ as a function of δ in the range $[0, 2\pi]$ ($\theta = 0.75$ (red), 1 (green), 1.25 (blue))

The last diagram that contributes at this order is given by

$$(f) = \int_0^\theta \int_0^\theta \frac{\lambda(\cos(\delta)(\cosh(t_1)\cosh(t_2) - 1) + \sinh(t_1)\sinh(t_2))}{16\pi^2(\sinh(t_1)\sinh(t_2)\cos(\delta) + \cosh(t_1)\cosh(t_2) - 1)} dt_1 dt_2 \quad (4.25)$$

Then we sum all the different contributions and evaluate numerically the result for three configurations with $\theta = 0.75, 1, 1.25$. Defining the function $R(\delta, \theta_0)$ as

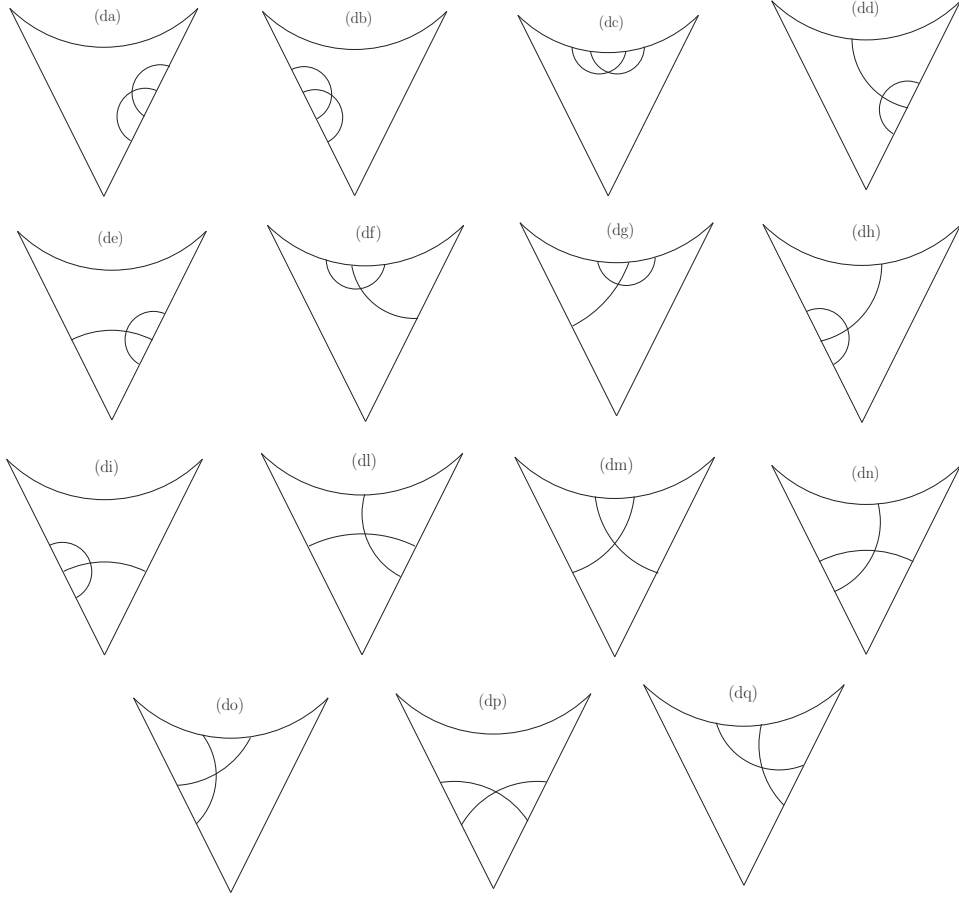
$$\langle W \rangle = -\lambda R(\delta, \theta_0) \quad (4.26)$$

we can plot it as a function of the opening angle δ , finding the result shown in figure 4.3. Consistently with the calculation of [90], if we note that for our particular choice of the shape the area enclosed by the loop is given by

$$\mathcal{A} = 2\delta \sinh^2\left(\frac{\theta}{2}\right) \quad (4.27)$$

we can see that $R[\delta, \theta_0]$ is in agreement (up to relative error of 10^{-7}) with the YM_2 prediction (the first order expansion of the matrix model 4.1) after the rescaling of the coupling constant (4.2)

$$\langle W \rangle_{YM_2} = 1 - \frac{\mathcal{A}(\mathcal{A} + 4\pi)\lambda}{32\pi^2} = 1 - \frac{\lambda \delta \sinh^2\left(\frac{\theta}{2}\right) (2\delta \sinh^2\left(\frac{\theta}{2}\right) + 4\pi)}{16\pi^2}. \quad (4.28)$$

Figure 4.4: *Double-exchange* diagrams

The next step is to take into account the double-exchange diagrams, as first contributions at order g^4 . We shall not consider the abelian part since these diagrams are given by $1/2$ the square of the contributions of order g^2 (see the previous section) but we shall focus on the maximally non-abelian part. The relevant diagrams are depicted in figure 4.4.

To begin, we consider the equivalent diagrams (da) and (db) of fig.4.4. Their value is given by

$$(da) + (db) = 2(db) = - \int_0^\theta dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \frac{g^4(N^2 - 1)}{128\pi^4} = - \frac{g^4(N^2 - 1)\theta^4}{3072\pi^4}. \quad (4.29)$$

For the diagram (dc) we can write

$$(dc) = - \int_{\pi-\delta}^{\pi} dt_1 \int_{\pi-\delta}^{t_1} dt_2 \int_{\pi-\delta}^{t_2} dt_3 \int_{\pi-\delta}^{t_3} dt_4 \frac{g^4(N^2-1) \sinh^4(\theta)}{256\pi^4} = - \frac{g^4(N^2-1) \sinh^4(\theta) \delta^4}{6144\pi^4}. \quad (4.30)$$

Consider now the diagram (dd) (equal to (dh)); we have

$$(dd) = (dh) = - \int_{\pi-\delta}^{\pi} dt_4 \int_{-\theta}^0 dt_1 \int_{-\theta}^{t_1} dt_2 \int_{-\theta}^{t_2} dt_3 \frac{g^4(N^2-1) \sin(t_2) \sinh^3(\theta) (\cosh(\theta) - \cosh(t_4))}{256\pi^4 (-\cos(t_2) \sinh(t_4) \sinh(\theta) + \cosh(t_4) \cosh(\theta) - 1)} \quad (4.31)$$

The integrations over t_1 and t_3 are trivial and can be performed analytically to obtain

$$(dd) = (dh) = \int_0^{\delta} dt_4 \int_0^{\theta} dt_2 \frac{g^4(N^2-1) t_2 (\theta - t_2) \sin(t_4) \sinh(\theta) (\cosh(t_2) - \cosh(\theta))}{256\pi^4 (-\sinh(t_2) \cos(t_4) \sinh(\theta) + \cosh(t_2) \cosh(\theta) - 1)}, \quad (4.32)$$

which can be evaluated numerically. Diagrams (de) and (df) , respectively equivalent to (dg) and (di) , are of the same type of (dd) . Indeed two of the four integrations over the circuit variables can be done easily remaining with the integrals

$$(de) = (di) = \int_0^{\theta} dt_2 \int_0^{\theta} dt_4 \frac{g^4(N^2-1) t_2 (\theta - t_2) (\cos(\delta) (\cosh(t_2) \cosh(t_4) - 1) - \sinh(t_2) \sinh(t_4))}{256\pi^4 (-\sinh(t_2) \sinh(t_4) \cos(\delta) + \cosh(t_2) \cosh(t_4) - 1)} \quad (4.33)$$

$$(df) = (dg) = - \int_0^{\delta} dt_4 \int_0^{\theta} dt_2 \frac{g^4(N^2-1) t_2 \delta (t_2 \delta - \delta) \sinh^3(\theta) \sin(t_2 \delta) (\cosh(t_4 \theta) - \cosh(\theta))}{256\pi^4 (-\sinh(\theta) \cos(t_2 \delta) \sinh(t_4 \theta) + \cosh(\theta) \cosh(t_4 \theta) - 1)} \quad (4.34)$$

Other double exchange diagrams are (do) , (dl) (respectively equal to (dq) , (dn)), (dp) and (dm) . They can be written as

$$(do) = (dq) = - \int_0^{\theta} dt_1 \int_0^{t_1} dt_2 \int_0^{\delta} dt_3 \int_0^{t_3} dt_4 \frac{g^4(N^2-1) \sin(t_3) \sin(t_4) \sinh^2(\theta)}{(2 \sinh(t_1) \cos(t_3) \sinh(\theta) - 2 \cosh(t_1) \cosh(\theta) + 2) (\cosh(\theta) - \cosh(t_1)) (\cosh(\theta) - \cosh(t_2))} \frac{1}{64\pi^4 (2 \sinh(t_2) \cos(t_4) \sinh(\theta) - 2 \cosh(t_2) \cosh(\theta) + 2)} \quad (4.35)$$

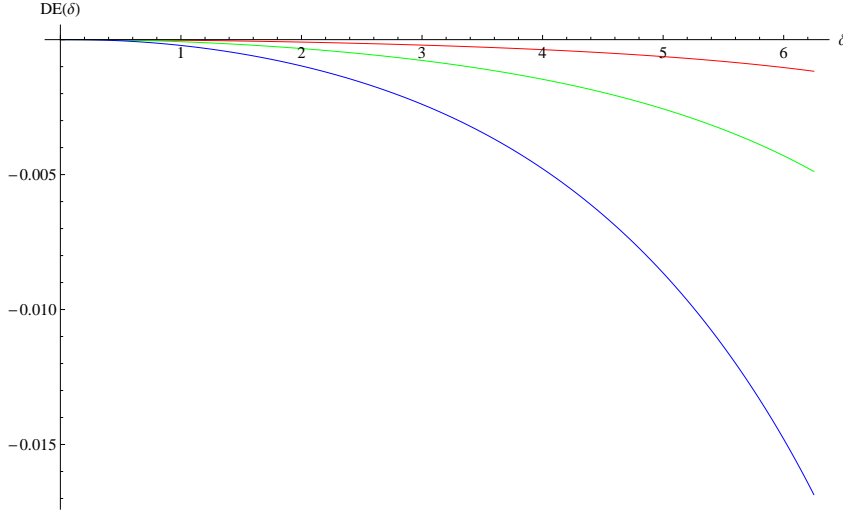


Figure 4.5: *Double-Exchange* diagrams as function of δ in the range $[0, 2\pi]$, ($\theta = 0.75$ (red), 1 (green), 1.25 (blue))

$$(dl) = (dn) = - \int_0^\theta dt_4 \int_0^\theta dt_1 \int_0^{t_1} dt_2 \int_0^\delta dt_3 \frac{g^4(N^2 - 1) \sin(t_3) \sinh(\theta) (\cosh(t_1) - \cosh(\theta))}{(-\sinh(t_1) \cos(t_3) \sinh(\theta) + \cosh(t_1) \cosh(\theta) - 1)} \frac{(\cosh(t_2) \cosh(t_4) \cos(\delta) - \sinh(t_2) \sinh(t_4) - \cos(\delta))}{256\pi^4 (-\sinh(t_2) \sinh(t_4) \cos(\delta) + \cosh(t_2) \cosh(t_4) - 1)} \quad (4.36)$$

$$(dp) = - \int_0^\theta dt_1 \int_0^{t_1} dt_2 \int_0^\delta dt_4 \int_0^{t_4} dt_3 \frac{g^4(N^2 - 1) (\cosh(t_1) \cosh(t_3) \cos(\delta) - \sinh(t_1) \sinh(t_3) - \cos(\delta))}{(-\sinh(t_1) \sinh(t_3) \cos(\delta) + \cosh(t_1) \cosh(t_3) - 1)} \frac{(\cosh(t_2) \cosh(t_4) \cos(\delta) - \sinh(t_2) \sinh(t_4) - \cos(\delta))}{256\pi^4 (-\sinh(t_2) \sinh(t_4) \cos(\delta) + \cosh(t_2) \cosh(t_4) - 1)} \quad (4.37)$$

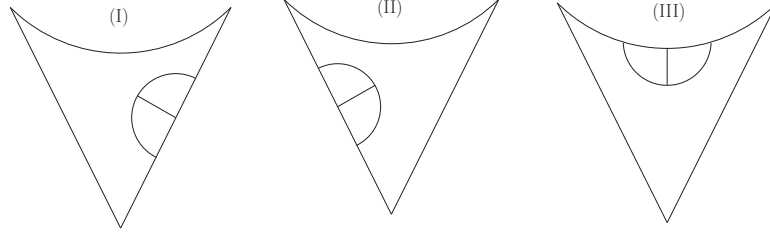
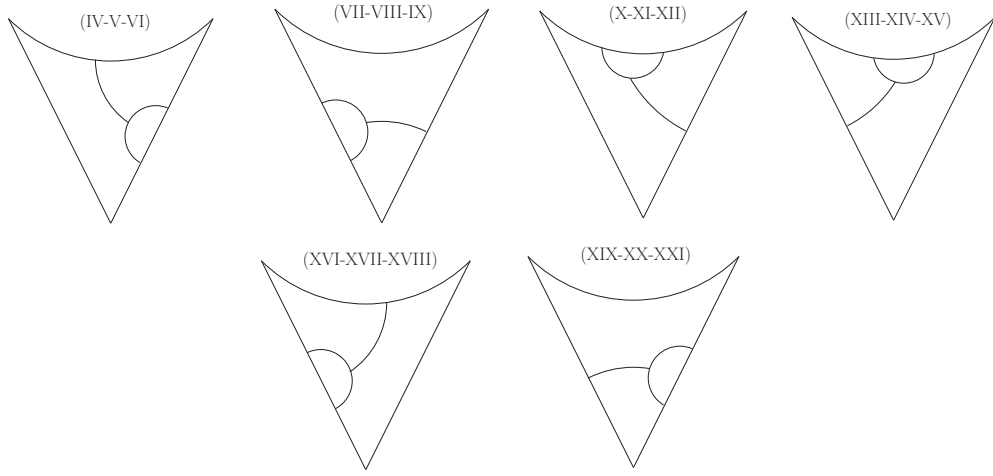
$$(dm) = - \int_0^\theta dt_2 \int_0^{t_2} dt_1 \int_0^\theta dt_4 \int_0^\theta dt_3 \frac{g^4(N^2 - 1) \sin(t_2) \sinh^2(\theta) \sin(t_1 - \delta)}{(-\cos(t_2) \sinh(t_3) \sinh(\theta) + \cosh(t_3) \cosh(\theta) - 1)} \frac{(\cosh(t_3) - \cosh(\theta)) (\cosh(\theta) - \cosh(t_4))}{256\pi^4 (-\sinh(t_4) \sinh(\theta) \cos(t_1 - \delta) + \cosh(t_4) \cosh(\theta) - 1)} \quad (4.38)$$

Summing up all all double-exchange diagrams and defining the function $DE[\delta, \theta_0]$ (θ_0 fixed) as

$$\text{Doubled-Exchange} = g^4(N^2 - 1) DE[\delta, \theta_0] \quad (4.39)$$

we can plot the result in figure (4.5). Having evaluated all ladder diagrams, we move to analyzing the more complicated interaction diagrams. In order to write these integrals systematically, we distinguish three different cases:

- (A) when the three legs of the vertex are attached to the same edge of the circuit (see fig.4.6, diagrams I-III)
- (B) when two of the three legs of the vertex are attached to the same edge of the circuit (see fig.4.7, diagrams IV-XXI)

Figure 4.6: *Interactions* diagrams - (A) typeFigure 4.7: *Interactions* diagrams - (B) type

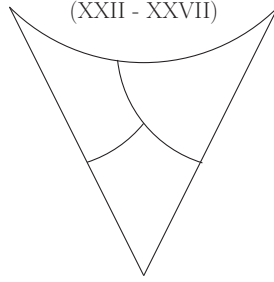
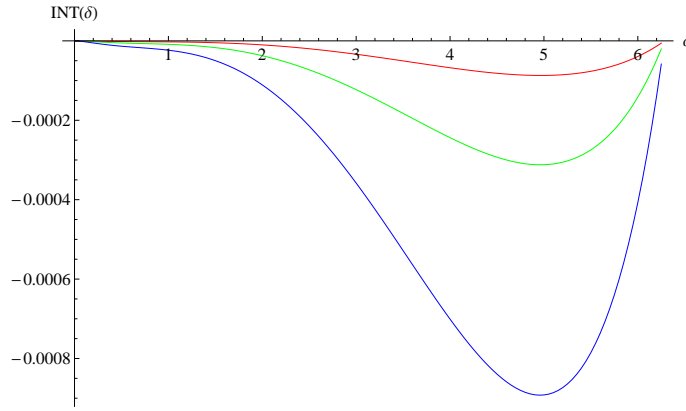
- (C) when all legs of the vertex are attached on different edge of the circuits (see figs.4.8, diagrams XXII-XXVII).

The diagrams belonging to the family (A) vanish, in fact the integral

$$\frac{g^4(N^2 - 1)}{64\pi^4} \int_0^\theta dt_1 \int_0^\theta dt_2 \int_0^\theta dt_3 \epsilon(t_1, t_2, t_3) \frac{\coth(\frac{t_2 - t_3}{2}) \log\left(\frac{1 - \coth(t_1 - t_2)}{1 - \coth(t_1 - t_3)}\right)}{512\pi^4}. \quad (4.40)$$

is equal to zero because the integrand is antisymmetric in the interchange (t_2, t_3) , while the integration region is symmetric. We are finally left with the diagrams belonging to the family (B) and (C), in total twenty-four non vanishing diagrams.

Here we only report the general formula for the total interactions, self-energy plus three vertex contributions and we remaind the reader to the *appendix G* where all interaction integrals are written explicitly. In summary we have to evaluated the formula

Figure 4.8: *Interactions* diagrams - (C) typeFigure 4.9: Plot of all-interaction diagrams as a function of δ in the range $[0, 2\pi]$, ($\theta = 0.75$ (red), 1 (green), 1.25 (blue))

$$\begin{aligned} \mathcal{I}_{\text{tot}} = & \frac{g^4(N^2 - 1)}{128\pi^4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) \frac{(\omega_{x_3} \cdot \omega_{x_1} - x_1 \cdot x_3)}{(x_3 - x_1)^2} \times \\ & \times \frac{(x_3 - x_2) \cdot \dot{x}_2}{(x_3 - x_2)^2} \log \left(\frac{(x_2 - x_1)^2}{(x_3 - x_1)^2} \right). \end{aligned} \quad (4.41)$$

for the case *B* in which two of x_1, x_2, x_3 variables are on the same edge and in the case *C* where each variable (x_1, x_2, x_3) is on a different edge. After writing down all the integrals, they can be evaluated numerically with Mathematica. Defining the function $INT[\delta, \theta_0]$ as

$$\text{Total Interactions} = g^4(N^2 - 1)INT[\delta, \theta_0] \quad (4.42)$$

it can be plotted in the fig.(4.9). At this point we can collect all the numerical results, doubled-exchange plus interaction diagrams, to obtain the maximal non-abelian contribution at order g^4 . If we define

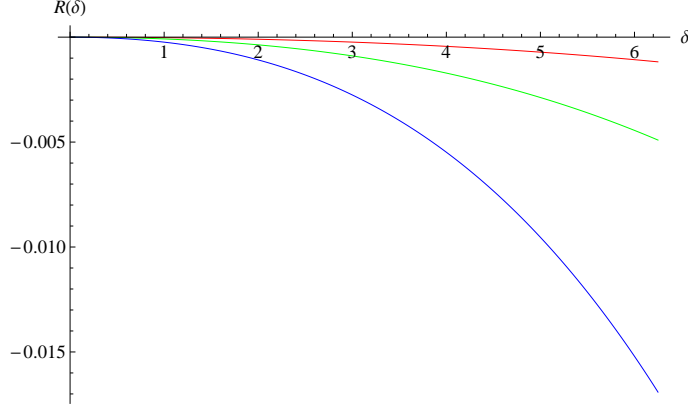


Figure 4.10: Plot of $R[\delta] = \text{INT}[\delta] + \text{DE}[\delta]$ as a function of δ in the range $[0, 2\pi]$, ($\theta = 0.75$ (red), 1 (green), 1.25 (blue))

$$R[\delta, \theta_0] = \text{INT}[\delta, \theta_0] + \text{DE}[\delta, \theta_0] \quad (4.43)$$

we plot $R[\delta, \theta_0]$ as a function of the opening angle δ (for θ_0 fixed) in fig.(4.10). In the next steps we perform a polynomial fit of the numerical result $R[\delta]$; we find that this function is well fitted by $-\frac{1}{6}$ the square of the contribution to the single exchange

$$R[\delta, \theta] = -\frac{1}{6} \left(\frac{\delta \sinh^2 \left(\frac{\theta}{2} \right) (2\delta \sinh^2 \left(\frac{\theta}{2} \right) + 4\pi)}{16\pi^2} \right)^2. \quad (4.44)$$

Indeed the relative error between the numerical calculation and the polynomial that fits it is less than 10^{-5} in the whole range of the opening angle ($0 < \delta < 2\pi$). If we add also the abelian part that is $\frac{1}{2}$ of the square of the single exchange contribution, we obtain

$$\begin{aligned} \langle W \rangle_{g^4} = & \left(\frac{g^4 N^2}{2} - \frac{g^4 (N^2 - 1)}{6} \right) \left(\frac{\delta \sinh^2 \left(\frac{\theta}{2} \right) (2\delta \sinh^2 \left(\frac{\theta}{2} \right) + 4\pi)}{16\pi^2} \right)^2 \\ & + \frac{2N^2 + 1}{6} \left(\frac{\delta \sinh^2 \left(\frac{\theta}{2} \right) (2\delta \sinh^2 \left(\frac{\theta}{2} \right) + 4\pi)}{16\pi^2} \right)^2 \end{aligned} \quad (4.45)$$

If we use eq.(4.27), it is easy to see that previous equation coincides precisely with the result of the Wilson loops computation in the zero instanton sector of the Yang Mills 2d on the H_2 . In fact the perturbative expansion of the matrix model (4.1) (up to order g^4)

$$\langle W \rangle = 1 - \frac{\mathcal{A}(\mathcal{A} + 4\pi)\lambda}{8\pi} + \frac{\mathcal{A}^2(\mathcal{A} + 4\pi)^2 g^4 (2N^2 + 1)}{384\pi^2} \quad (4.46)$$

exactly coincides with (4.45) after the usual rescaling of the coupling constant.

Chapter 5

Correlator of Wilson Loops

5.1 Introduction

In the present chapter we study in detail the correlators of two Wilson loops on S^2 introduced in the previous chapters. A similar analysis for the correlators of Maldacena-Wilson loops has already been done in [93, 94, 95] where these operators have been studied both at strong and weak coupling. First of all, after a brief review of the supersymmetric properties of the correlators, we extend the conjecture analyzed in the previous sections; we derive a general formula valid for any coupling constant g and any value of N for correlators of BPS Wilson loops with arbitrary contours on S^2 in terms of the multi-matrix model governing the zero instanton expansion of YM_2 .

To check our conjecture we report some interesting test. Perturbatively a highly non-trivial computation in $\mathcal{N} = 4$ SYM for the correlator of two latitude has been performed (up to g^6 order) finding for two particular configurations a numerical agreement with the matrix model prediction. Afterwards we investigate analytically the limit where one of the two latitudes shrinks to zero size: since our non-perturbative formula is an order by order polynomial in the shrinking radius, the absence of logarithmic terms is a crucial test of the matrix representation. We find indeed the absence of leading logarithms in the shrinking radius, a quite non-trivial result, differing dramatically from the analogous computation of non-BPS correlators [94] where logs are present. Interestingly, by analyzing the OPE of the shrinking Wilson loop one can relate the absence of the logarithmic terms to the protection of a local operator which may be expressed as the trace of the square of a twisted field strength.

Finally we can take the large N and strong coupling limit and try to compare it to the $\mathcal{N} = 4$ correlators from the string side. In the limit where the two latitudes shrink to opposite poles on the sphere, this calculation reduces to the

semi-classical exchange of supergravity (SUGRA) modes between the two string worldsheets describing the Wilson loops at strong coupling. We find that at leading order in the large-separation limit, the matrix model result seems to capture the exchange of the SUGRA modes dual to a certain chiral primary operator. Other modes, dual to other protected operators present in the weak coupling OPE, has also been carefully included but while reproducing the geometrical dependence of the matrix model result, we find a mismatch in the numerical coefficients.

5.2 The conjectured matrix model description

In this subsection we try to argue that the correlator of two (or more) Wilson-loops of type (3.3) might be an exactly solvable quantity since it belongs to a topological sector of $\mathcal{N} = 4$. In particular we focus our attention on the problem of writing a general formula for the correlator of two Wilson-loops. The starting point is to recall that, as we have seen in the previous sections, the expectation value of one Wilson-loop appears to be computed by the matrix model governing the zero-instanton sector of YM_2 on the two sphere [59, 75, 76]. Since as we will show the single Wilson loop and the correlator generically share the same symmetries we expect that this equivalence also extends to the case of correlators. Therefore we conjecture that

$$\begin{aligned} &\text{The Correlator of two DGRT Wilson loops on } S^2 \\ &\quad \text{in } \mathcal{N} = 4 \text{ Supersymmetric Yang Mills theory} \\ &\quad \equiv \\ &\text{The Correlator of two Wilson loops on } S^2 \\ &\quad \text{in the zero instanton sector of } \text{YM}_2 \end{aligned}$$

The construction of the matrix model governing the zero instanton sector of YM_2 is quite simple since YM_2 is an almost topological theory (it is invariant under area-preserving diffeomorphisms) and its observables can be computed with the help of some simple *string-like* Feynman-rules [10]. For the present computation we need just three ingredients: the cylinder amplitude (*heat-kernel propagator*), the disc and the Feynman rule for the observable, *i.e.* the *Wilson loop*. The first quantity is represented in fig.(5.1) and is given by

$$\mathcal{K}(A; U_1, U_2) = \langle U_2 | e^{-\frac{g^2 A \Delta}{2}} | U_1 \rangle = \sum_R \chi_R(U_1) \chi_R^\dagger(U_2) e^{-\frac{g^2 A}{2} C_2(R)}, \quad (5.1)$$

where A is the area of the cylinder and the sum runs over all the representations R of $U(N)$. The amplitude also depends on the two holonomies U_1 and U_2 defined

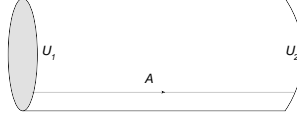


Figure 5.1: Propagator

on the two borders of the cylinder. There is a dual representation for the cylinder amplitude where the sum over representations is replaced with a sum over the instanton charges

$$\mathcal{K}(A; U_1, U_2) = \sum_{P \in S_N} \frac{(g^2 A)^{-N/2}}{J(\theta_i) J(\phi_i)} \sum_{\ell \in \mathbb{Z}^N} (-1)^{P+(N-1)\sum \ell_i} \exp \left(-\frac{1}{2g^2 A} \sum_{i=1}^N (\phi_i - \theta_{P(i)} + 2\pi i \ell_i)^2 \right), \quad (5.2)$$

where $\{e^{i\theta_i}\}$ and $\{e^{i\phi_i}\}$ are the eigenvalues of the matrices U_1 and U_2 respectively and

$$J(\theta_i) = \prod_{i \leq j} 2 \sin \left(\frac{\theta_i - \theta_j}{2} \right).$$

The disc is obtained from (5.2) by choosing one of the two holonomies to be trivial, namely equal to the identity. Finally, the insertion of a Wilson loop with winding number n is realized by introducing the factor $\text{Tr}(U^n)$ at the border of the cylinder. The amplitude for the correlator of two non-intersecting loops with winding numbers n_1 and n_2 is schematically represented in fig.(5.2), and the corresponding expression is given by the following two-matrix integral over the unitary matrices:

$$\begin{aligned} \tilde{\mathcal{W}}(A_1, A_2) &= \frac{1}{N^2} \int \mathcal{D}U_1 \mathcal{D}U_2 \text{Tr}(U_1^{n_1}) \text{Tr}(U_2^{n_2}) \mathcal{K}(A_1; 1, U_1) \mathcal{K}(A_3; U_1, U_2) \mathcal{K}(A_2; U_2, 1) = \\ &= \frac{1}{N^2} \sum_{P \in S_N} \sum_{\ell, m, s \in \mathbb{Z}^N} \int d^N \theta d^N \phi J^2(\theta_i) J^2(\phi_i) \left(\sum_{r,s=1}^N e^{in_1 \theta_r + in_2 \phi_s} \right) \times \\ &\quad \times \frac{(g^2 A_1)^{-\frac{N^2}{2}}}{J(\theta_i)} (-1)^{(N-1)\sum_i \ell_i} \Delta(\theta_i + 2\pi \ell_i) \exp \left(-\frac{1}{2g^2 A_1} \sum_{i=1}^N (\theta_i + 2\pi \ell_i)^2 \right) \times \\ &\quad \times \frac{(g^2 A_3)^{-N/2}}{J(\theta_i) J(\phi_i)} (-1)^{P+(N-1)\sum s_i} \exp \left(-\frac{1}{2g^2 A_3} \sum_{i=1}^N (\phi_i - \theta_{P(i)} + 2\pi i s_i)^2 \right) \\ &\quad \times \frac{(g^2 A_2)^{-\frac{N^2}{2}}}{J(\phi_i)} (-1)^{(N-1)\sum_j m_j} \Delta(\phi_j + 2\pi m_j) \exp \left(-\frac{1}{2g^2 A_2} \sum_{i=1}^N (\phi_i + 2\pi m_i)^2 \right), \end{aligned} \quad (5.3)$$

Δ being the Vandermonde determinant.

The amplitude $\tilde{\mathcal{W}}(A_1, A_2)$ is related to the true correlator by the relation $\tilde{\mathcal{W}}(A_1, A_2) = \mathcal{Z} \mathcal{W}(A_1, A_2)$, where \mathcal{Z} is the partition function of YM_2 on the sphere. We can extend the region of integration over the entire R^{2N} by means of the sum over ℓ and m and we can rewrite the above expression as

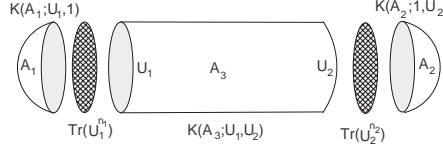


Figure 5.2: The string-like Feynman-diagram for the correlator of two Wilson-loops.

$$\begin{aligned} \tilde{W}(A_1, A_2) &= \frac{(g^4 A_1 A_2)^{-\frac{N^2}{2}} (g^2 A_3)^{-\frac{N}{2}}}{N^2} \sum_{P \in S_N} \sum_{s \in \mathbb{Z}^N} (-1)^{P+(N-1)\sum s_i} \int_{\mathbb{R}^{2N}} d^N \theta d^N \phi \left(\sum_{r,s=1}^N e^{in_1 \theta_r + in_2 \phi_s} \right) \times \\ &\times \Delta(\theta_i) \Delta(\phi_i) \exp \left(-\frac{1}{2g^2 A_1} \sum_{i=1}^N \theta_i^2 - \frac{1}{2g^2 A_3} \sum_{i=1}^N (\phi_i - \theta_{P(i)} + 2\pi i s_i)^2 - \frac{1}{2g^2 A_2} \sum_{i=1}^N \phi_i^2 \right). \end{aligned} \quad (5.4)$$

The result (5.4) is the exact amplitude and it contains all instantonic corrections. To single out the zero-instanton sector of this amplitude it is sufficient to consider the case where all instanton numbers s_i vanish. If we introduce the diagonal matrices $\Theta = \text{diag}(\theta_1, \dots, \theta_N)$ and $\Phi = \text{diag}(\phi_1, \dots, \phi_N)$, using the Itzykson-Zuber integration formula and defining the hermitian matrices $V_1 = U^{-1} \Theta U$ and $V_2 = V \Phi V^{-1}$, we can recast the original integral as the following hermitian two matrix model for the correlator of two Wilson loops

$$\begin{aligned} W(A_1, A_2) &= \frac{1}{C_N N^2} \int DV_1 DV_2 e^{-\frac{A_1+A_3}{2g^2 A_1 A_3} \text{Tr}(V_1^2) - \frac{A_2+A_3}{2g^2 A_2 A_3} \text{Tr}(V_2^2) + \frac{1}{g^2 A_3} \text{Tr}(V_1 V_2)} \text{Tr}(e^{in_1 V_1}) \text{Tr}(e^{in_2 V_2}) = \\ &= \frac{1}{C_N N^2} \int DV_1 DV_2 e^{-\frac{1}{2g^2 A_1} \text{Tr}(V_1^2) - \frac{1}{2g^2 A_2} \text{Tr}(V_2^2) - \frac{1}{2g^2 A_3} \text{Tr}((V_1 - V_2)^2)} \text{Tr}(e^{in_1 V_1}) \text{Tr}(e^{in_2 V_2}), \end{aligned} \quad (5.5)$$

where the normalization is chosen to be

$$C_N = \int DV_1 DV_2 e^{-\frac{A_1+A_3}{2g^2 A_1 A_3} \text{Tr}(V_1^2) - \frac{A_2+A_3}{2g^2 A_2 A_3} \text{Tr}(V_2^2) + \frac{1}{g^2 A_3} \text{Tr}(V_1 V_2)}. \quad (5.6)$$

Actually, in the sector $s_i = 0$ of (5.4), the angular integration can be performed by means of an expansion in terms of Hermite polynomials and by exploiting the relation between integrals over Hermite polynomials and Laguerre polynomials. Then one finds the following finite N closed expression for the connected correlator

$$\begin{aligned} W(A_1, A_2) - W(A_1)W(A_2) &= \\ &= \frac{1}{N^2} e^{-\frac{(A_1 A_2 (n_1+n_2)^2 + A_3 (n_1^2 A_1 + n_2^2 A_2)) g^2}{2A}} L_{N-1}^1 \left(\frac{g^2 (A_3 n_1 + A_2 (n_1 + n_2)) (A_1 (n_1 + n_2) + A_3 n_2)}{A} \right) + \\ &- \frac{1}{N^2} e^{-\frac{(A_1 (A_2 + A_3) n_1^2 + A_2 (A_1 + A_3) n_2^2) g^2}{2A}} \times \\ &\times \sum_{i_1, i_2=1}^N \left(-\frac{g^2 n_1 n_2 A_1 A_2}{A} \right)^{i_2-i_1} \frac{(i_1-1)!}{(i_2-1)!} L_{i_1-1}^{i_2-i_1} \left(\frac{g^2 n_2^2 A_2 (A_3 + A_1)}{A} \right) L_{i_1-1}^{i_2-i_1} \left(\frac{g^2 n_1^2 A_1 (A_3 + A_2)}{A} \right), \end{aligned} \quad (5.7)$$

where $A = A_1 + A_2 + A_3$ is the total area of the sphere. For small g this expression can be expanded in a power series and one finds

$$\begin{aligned}
W(A_1, A_2) - W(A_1)W(A_2) = & -\frac{A_1 A_2 g^2 n_1 n_2}{NA} + \\
& + \frac{A_1 A_2 (A_1 A_2 (n_1^2 + n_2^2 + n_1 n_2) + A_3 (A_1 n_1^2 + A_2 n_2^2)) g^4 n_1 n_2}{2A^2} + \\
& - g^6 n_1 n_2 \left(\frac{A_1^3 A_2 (A_2 + A_3)^2 (2N^3 + N) n_1^4}{24A^3 N^2} + \frac{A_1^3 A_2^2 (A_2 + A_3) (2N^3 + N) n_2 n_1^3}{12A^3 N^2} + \right. \\
& + \frac{A_1^2 A_2^2 (3A_3 (A_2 + A_3) N^2 + A_1 (3A_3 N^2 + A_2 (4N^2 + 1))) n_2^2 n_1^2}{12A^3 N} + \\
& \left. + \frac{A_1^2 A_2^3 (A_1 + A_3) (2N^3 + N) n_2^3 n_1}{12A^3 N^2} + \frac{A_1 A_2^3 (A_1 + A_3)^2 (2N^3 + N) n_2^4}{24A^3 N^2} \right) + O(g^7)
\end{aligned} \tag{5.8}$$

In the following subsection we compare this perturbative result (5.8) with the actual computation in $\mathcal{N} = 4$ SYM. After performing the standard redefinition $g^2 \mapsto -g^2/A$ and setting $n_1 = n_2 = 1$, we find a complete agreement up to order g^4 and numerical evidences at g^6 order. Notice, moreover, that the agreement with YM_2 demands the absence of logarithmic singularities when the area of one of the loops is small, to all orders in perturbation theory. Our result of subsec.(5.5) is consistent with this prediction.

In order to analyse the large N limit, we can write a simple compact representation for the connected correlator in $\mathcal{N} = 4$ SYM by exploiting a contour representation of the Laguerre polynomials

$$W(A_1, A_2) - W(A_1)W(A_2) = \frac{n_1 n_2}{N^2} \int_{C_1} \frac{dw_1}{2\pi i} \int_{C_2} \frac{dw_2}{2\pi i} \frac{e^{w_1 + w_2 + \frac{\lambda(\tilde{A}_1 A_1 w_2 n_1^2 + \tilde{A}_2 A_2 n_2^2 w_1)}{A^2 w_2 w_1}}}{\left(\tilde{A}_2 n_2 w_1 - A_1 n_1 w_2\right)^2} \tilde{A}_2 A_1, \tag{5.9}$$

where $\tilde{A}_1 = A - A_1$ and $\tilde{A}_2 = A - A_2$. This expression can be computed as an infinite series of Bessel functions. We limit our attention to the case $n_1 = n_2 = 1$ and are actually interested in the normalized correlator, which is given by

$$\frac{W_{conn.}}{W_1 W_2} = \frac{\lambda}{N^2 A^2} \tilde{A}_1 \tilde{A}_2 \sum_{k=1}^{\infty} k \left(\sqrt{\frac{A_1 A_2}{\tilde{A}_1 \tilde{A}_2}} \right)^{k+1} \frac{I_k \left(2\sqrt{\frac{\lambda A_2 \tilde{A}_2}{A^2}} \right) I_k \left(2\sqrt{\frac{\lambda A_1 \tilde{A}_1}{A^2}} \right)}{I_1 \left(2\sqrt{\frac{\lambda A_2 \tilde{A}_2}{A^2}} \right) I_1 \left(2\sqrt{\frac{\lambda A_1 \tilde{A}_1}{A^2}} \right)}. \tag{5.10}$$

In the subsection (5.6) we will be interested in comparing this result with the strong coupling prediction of super-gravity. For this reason, we have to expand the above result for large λ . This can easily be done by recalling that

$$\frac{I_k(z)}{I_1(z)} = 1 + O\left(\frac{1}{z}\right). \tag{5.11}$$

Then the correlator in the strong coupling regime becomes

$$\frac{W_{conn.}}{W_1 W_2} \sim \frac{\lambda}{N^2} \frac{\tilde{A}_1 \tilde{A}_2}{A^2} \left[\frac{A_1 A_2}{\tilde{A}_1 \tilde{A}_2} + 2 \left(\sqrt{\frac{A_1 A_2}{\tilde{A}_1 \tilde{A}_2}} \right)^3 + \dots \right]. \quad (5.12)$$

The first term in the expansion corresponds to the $U(1)$ factor present in $U(N)$ and we shall drop it since it is not generally considered in the super-gravity analysis. The first non trivial term which can be compared with super-gravity is the second one.

5.3 Correlators in $\mathcal{N}=4$ SYM : Perturbative Computation

As we have already shown in section (3.1), demanding the vanishing of the supersymmetric variation of the Wilson loop on a latitude at polar angle θ_0 one finds two relations

$$\cos \theta_0 (\gamma_{12} + \rho_{12}) \epsilon_1 = 0 \quad (5.13)$$

$$\rho^3 \gamma^5 \epsilon_0 = [i \gamma_{12} + \gamma_3 \rho^2 \gamma^5 \cos \theta (\gamma_{23} + \rho_{23})] \epsilon_1. \quad (5.14)$$

It is clear that each of them reduce the supersymmetry by half, and therefore a single latitude is 1/4 BPS. We will be mainly interested in the correlator of two such Wilson loops, as shown in figure (5.3). The first relation (5.13) is shared between two such latitudes, whereas the second is clearly not. Thus two latitudes are collectively 1/8 BPS, each sharing half of their individual supersymmetry. For a particular case in which one of the two latitude is at the equator the system is 1/4 BPS.

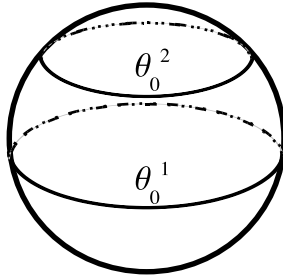


Figure 5.3: Two Wilson loops given by latitudes at polar angle θ_0^1 and θ_0^2

In the following we perform a perturbative analysis up to order g^6 for the connected correlator $\mathcal{W}(C_1, C_2) \equiv W(C_1, C_2) - W(C_1)W(C_2)$ of two latitudes in the case that the gauge group is $U(N)$.

g^2 and g^4 computation To begin with, we shall consider the g^2 diagram depicted in fig. 5.4 [Notice that this contribution would be absent in a $SU(N)$ theory].

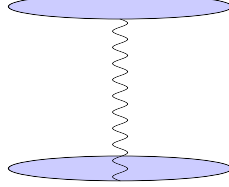


Figure 5.4: g^2 diagram

In order to carry out the computation, we parameterize the two circuits using polar coordinates

$$\begin{aligned} x_\mu^{C_1} &= R(0, \sin \theta_1 \cos \tau, \sin \theta_1 \sin \tau, \cos \theta_1) \\ y_\mu^{C_2} &= R(0, \sin \theta_2 \cos \sigma, \sin \theta_2 \sin \sigma, \cos \theta_2), \end{aligned} \quad (5.15)$$

and define the effective propagator $\Delta_{C_1 C_2}(\tau, \sigma)$ connecting the two loops

$$\Delta_{C_1 C_2}(\tau, \sigma) = \frac{2}{N} \langle \text{Tr}(\mathcal{A})(\tau) \text{Tr}(\mathcal{A})(\sigma) \rangle_0 = -\frac{\sin \theta_1 \sin \theta_2 (\cos(\tau - \sigma) (\cos \theta_1 \cos \theta_2 - 1) + \sin \theta_1 \sin \theta_2)}{8\pi^2 (\cos \theta_1 \cos \theta_2 + \cos(\tau - \sigma) \sin \theta_1 \sin \theta_2 - 1)}$$

where \mathcal{A} denotes the effective field $iA_\mu(x)\dot{x}^\mu + \Theta_I \Phi^I(x)|\dot{x}|$. Then the g^2 -contribution is given by

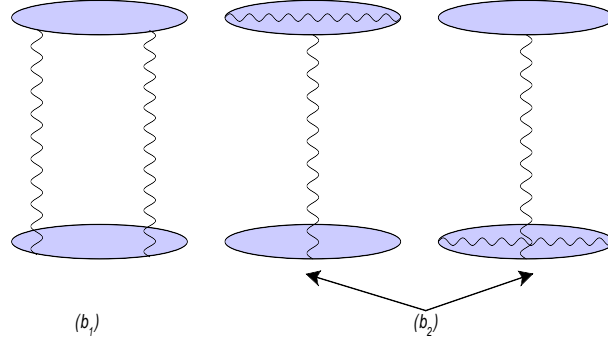
$$\langle \mathcal{W}(C_1, C_2) \rangle_{g^2} = \frac{g^2}{2N} \int_0^{2\pi} d\tau d\sigma \Delta_{C_1 C_2}(\tau, \sigma) = -\frac{\lambda}{N^2} \frac{A_1 A_2}{A^2} \quad (\lambda \equiv g^2 N), \quad (5.16)$$

where A is the total area of the sphere, and A_1 and A_2 are the areas enclosed by the two Wilson-loops given by

$$\frac{A_1}{A} = \frac{2\pi(1 - \cos \theta_1)}{4\pi} = \sin^2 \frac{\theta_1}{2} \quad \frac{A_2}{A} = \frac{2\pi(1 + \cos \theta_2)}{4\pi} = \cos^2 \frac{\theta_2}{2}. \quad (5.17)$$

At order g^4 , we have to consider the diagrams in fig.(5.5). First, we shall consider the contribution $S_{g^2-g^2}$ due to diagram (b_1) . Its evaluation reduces to the following integral over the circuits

$$\begin{aligned} S_{g^2-g^2} &= \frac{g^4}{16} \int_0^{2\pi} d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 \left[\Delta_{C_1 C_2}(\tau_1, \sigma_1) \Delta_{C_1 C_2}(\tau_2, \sigma_2) + \Delta_{C_1 C_2}(\tau_1, \sigma_2) \Delta_{C_1 C_2}(\tau_2, \sigma_1) \right] = \\ &= \frac{g^4}{8} \left[\int_0^{2\pi} d\tau_1 d\sigma_1 \Delta_{C_1 C_2}(\tau_1, \sigma_1) \right]^2 = \frac{g^4}{8} \left[2 \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right]^2 = \frac{\lambda^2}{2N^2} \frac{A_1^2 A_2^2}{A^4}. \end{aligned} \quad (5.18)$$

Figure 5.5: g^4 diagrams

Next we shall consider the contribution S_{g-g^3} due to the two diagrams (b_2) . The sum of the two diagrams yields

$$\begin{aligned}
 S_{g-g^3} &= \frac{g^4}{4!} \oint_{C_1} d\tau_1 \oint_{C_2} d\sigma_1 d\sigma_2 d\sigma_3 (\Delta_{C_1 C_2}(\tau_1, \sigma_1) \Delta_{C_2 C_2}(\sigma_2, \sigma_3) + \Delta_{C_1 C_2}(\tau_1, \sigma_2) \Delta_{C_2 C_2}(\sigma_1, \sigma_3) + \\
 &\quad + \Delta_{C_1 C_2}(\tau_1, \sigma_3) \Delta_{C_2 C_2}(\sigma_1, \sigma_2)) + (C_1 \leftrightarrow C_2) = \\
 &= \frac{g^4}{16} \left(2 \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) (\sin^2 \theta_1 + \sin^2 \theta_2) = \frac{\lambda^2}{2N^2 A^4} A_1 A_2 (A_1 A_3 + A_2 A_3 + 2A_1 A_2),
 \end{aligned} \tag{5.19}$$

where $\Delta_{C_2 C_2}(\sigma_i, \sigma_j) = \frac{\sin^2 \theta_2}{8\pi^2}$ and $A_3 = A - A_1 - A_2$. If we sum all the contributions at order g^4 to the one at order g^2 , the total result is

$$\langle \mathcal{W}(C_1, C_2) \rangle = -\frac{\lambda A_1 A_2}{N^2 A} + \frac{\lambda^2}{2N^2 A^4} A_1 A_2 (A_1 A_3 + A_2 A_3 + 3A_1 A_2). \tag{5.20}$$

that exactly reproduce the matrix model prediction (5.8) with $n_1 = n_2 = 1$.

g^6 computation We now come to considering the g^6 contributions. Since, at this order, the $\mathcal{N} = 4$ SYM interactions will start contribute, a complete analytic evaluation of all the relevant integrals is out of reach. However we will write compact formulas which can be used in the numerical evaluation. To do the numerical analysis two explicit configurations will be chosen (see (fig.5.3)):

- ◇ SYMMETRIC CASE: The two latitudes are located at opposite positions with respect to the equator of the 2-sphere, namely one at $\theta = \delta$ and the other at $\theta = \pi - \delta$, where θ denotes the standard polar coordinate on S^2 .
- ◇ ASYMMETRIC CASE: The first latitude is fixed and it is chosen to be the equator of S^2 , while the second latitude is free to move ($\theta = \delta$ with $0 \leq \delta \leq \pi$).

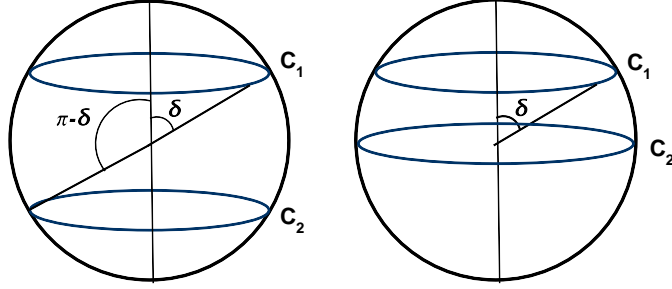
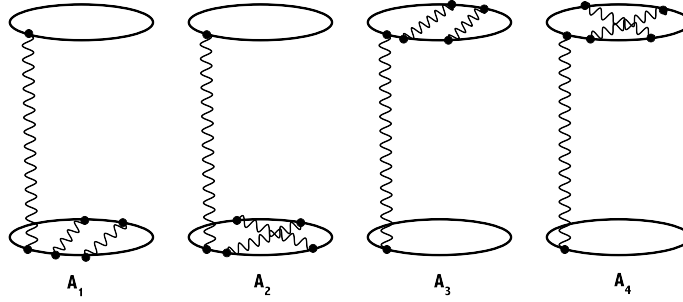


Figure 5.6: Symmetric configuration and Asymmetric one

Figure 5.7: The four diagrams $g \cdot g^5$

g^6 Ladder diagrams To begin with, we shall consider all the diagrams which do not contain interactions. They can be naturally split into three families characterized by the number of field insertions at each latitude. Therefore, at order g^6 one has to consider the following possibilities: $g \cdot g^5$, $g^2 \cdot g^4$ and $g^3 \cdot g^3$.

$g \cdot g^5$: We have four diagrams with only one propagator insertion in one of the two latitudes and we have schematically listed them in fig.(5.7). Notice that the third and the fourth diagram can be obtained from the first two by exchanging the two latitudes ($C_1 \leftrightarrow C_2$) and thus we have really to compute only two diagrams. In the following we shall denote with t the angular parameter running over the latitude C_1 and with s , the one spanning the second latitude C_2 . Then the contribution of the diagrams in fig.(5.7) can be summarized as follows

$$\begin{aligned} \mathbf{g}^1 \cdot \mathbf{g}^5 = & \frac{g^6}{N^2} \text{P} \oint_{C_1, C_2} dt_1 \prod_{i=2}^6 ds_i \langle \text{Tr}[\mathbf{A}(t_1)] \text{Tr}[\mathbf{A}(s_2) \mathbf{A}(s_3) \mathbf{A}(s_4) \mathbf{A}(s_5) \mathbf{A}(s_6)] \rangle_0 + \\ & + (C_1 \leftrightarrow C_2), \end{aligned} \quad (5.21)$$

where the symbol P in front of the integral means that the integration over the s_i is ordered ($0 \leq s_6 \leq s_5 \leq s_4 \leq s_3 \leq s_2 \leq 2\pi$) and \mathcal{A} stands for the usual effective connection constructed out of the gauge potential and the scalars. In (5.21) the

vacuum expectation value is obviously taken in the free theory and by expanding it in terms of free propagators we find

$$\begin{aligned} \mathbf{g}^1 \cdot \mathbf{g}^5 = & \frac{5g^6 N}{4} \text{P} \oint_{C_1 C_2} dt_1 \prod_{i=2}^6 ds_i \Delta_{12}(t_1, s_2) \Delta_{22}(s_3, s_4) \Delta_{22}(s_5, s_6) + \\ & + \frac{5g^6}{8N} \text{P} \oint_{C_1 C_2} dt_1 \prod_{i=2}^6 ds_i \Delta_{12}(t_1, s_2) \Delta_{22}(s_3, s_5) \Delta_{22}(s_4, s_6) + (C_1 \leftrightarrow C_2), \end{aligned} \quad (5.22)$$

where $\Delta_{12}(t_i, s_j)$ represents a propagator connecting the latitudes C_1 and C_2 , while $\Delta_{11}(t_i, t_j)$ and $\Delta_{22}(s_i, s_j)$ denote an internal exchange on C_1 and C_2 respectively. Their explicit expression, if we use the polar representation for our circuits ($C_1 = \{0, \sin \theta_1 \sin t, \sin \theta_1 \cos t, \cos \theta_1\}$, $C_2 = \{0, \sin \theta_2 \sin s, \sin \theta_2 \cos s, \cos \theta_2\}$), is given by

$$\begin{aligned} \Delta_{12}(t_i, s_j) &= \frac{\sin \theta_1 \sin \theta_2 ((\cos \theta_1 \cos \theta_2 - 1) \cos(t_i - s_j) + \sin(\theta_1) \sin(\theta_2))}{8\pi^2 (\sin \theta_1 \sin \theta_2 \cos(t_i - s_j) + \cos \theta_1 \cos \theta_2 - 1)} \\ \Delta_{11}(t_i, t_j) &= -\frac{\sin^2 \theta_1}{8\pi^2} \quad \Delta_{22}(s_i, s_j) = -\frac{\sin^2 \theta_2}{8\pi^2}. \end{aligned} \quad (5.23)$$

The integration over the two circuits can be easily performed in a closed form for two generic latitudes at $\theta = \theta_1$ and $\theta = \theta_2$ and we obtain the following compact expression

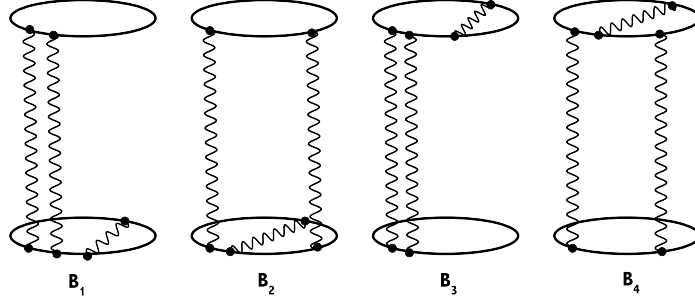
$$\mathbf{g}^1 \cdot \mathbf{g}^5 = \frac{g^6(N + 2N^3)}{24A^6N^2} (A_1^2(A_2 + A_3)^2 + A_2(A_1 + A_3)^2) A_1 A_2, \quad (5.24)$$

in terms of the area A_1 (A_2) enclosed by the circuit C_1 (C_2) and the area A_3 delimited by the two latitudes. For our choice of configurations, the above expression yields the following two results

$$\mathbf{g}^1 \cdot \mathbf{g}^5 = \begin{cases} \text{SYMMETRIC:} & \frac{g^6(1+2N^2)}{12N} \cos\left(\frac{\delta}{2}\right)^4 \sin\left(\frac{\delta}{2}\right)^8, \\ \text{ASYMMETRIC:} & \frac{g^6(1+2N^2)}{6144N} (11 - 4 \cos(2\delta) + \cos(4\delta)) \sin\left(\frac{\delta}{2}\right)^2. \end{cases} \quad (5.25)$$

$\boxed{\mathbf{g}^2 \cdot \mathbf{g}^4}$: Again we have four diagrams with two propagator insertions in one of the two latitudes and they are shown in the fig.(5.8). Using the same conventions introduced for the previous case, the contribution of the above diagrams reads

$$\begin{aligned} \mathbf{g}^2 \cdot \mathbf{g}^4 = & \frac{g^6}{N^2} \text{P} \oint_{C_1 C_2} dt_1 dt_2 \prod_{i=3}^6 ds_i (\text{Tr}[\mathbf{A}(t_1) \mathbf{A}(t_2)] \text{Tr}[\mathbf{A}(s_3) \mathbf{A}(s_4) \mathbf{A}(s_5) \mathbf{A}(s_6)])_0 + \\ & + (C_1 \leftrightarrow C_2). \end{aligned} \quad (5.26)$$

Figure 5.8: The four diagrams $g^2 \cdot g^4$

The symbol P denotes, this time, both the ordering in t -integration ($0 \leq t_2 \leq t_1 \leq 2\pi$) and in the s -integration ($0 \leq s_6 \leq s_5 \leq s_4 \leq s_3 \leq 2\pi$). If we expand the integrand of (5.26) in terms of free propagators, we obtain

$$\begin{aligned} \mathbf{g}^2 \cdot \mathbf{g}^4 &= \frac{g^6 N}{2} P \oint_{C_1 C_2} dt_1 dt_2 \prod_{i=3}^6 ds_i \Delta_{12}(t_1, s_3) \Delta_{12}(t_2, s_4) \Delta_{22}(s_5, s_6) + \\ &+ \frac{g^6}{4N} P \oint_{C_1 C_2} dt_1 dt_2 \prod_{i=3}^6 ds_i \Delta_{12}(s_1, s_3) \Delta_{12}(t_2, s_5) \Delta_{22}(s_4, s_6) + (C_1 \leftrightarrow C_2). \end{aligned} \quad (5.27)$$

The above expression can be evaluated for generic latitudes and yields

$$\mathbf{g}^2 \cdot \mathbf{g}^4 = \frac{g^6(N + 2N^3)}{12A^6 N^2} (A_1 A_3 + A_2 A_3 + 2A_1 A_2) A_1^2 A_2^2. \quad (5.28)$$

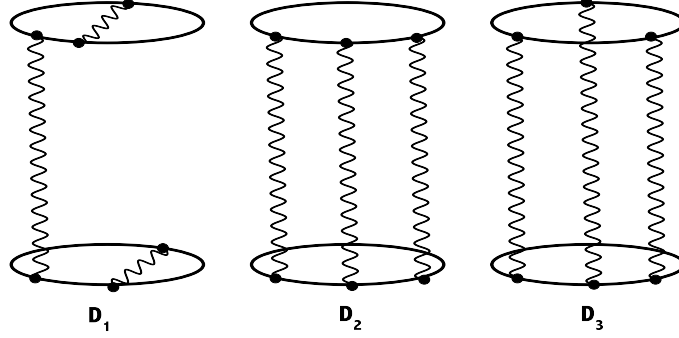
For our particular choice of the configurations this formula reduces to

$$\mathbf{g}^2 \cdot \mathbf{g}^4 = \begin{cases} \text{SYMMETRIC:} & \frac{g^6(1+2N^2)}{6N} \cos\left(\frac{\delta}{2}\right)^2 \sin\left(\frac{\delta}{2}\right)^{10} \\ \text{ASYMMETRIC:} & -\frac{g^6(1+2N^2)}{192N} \left(\cos(\delta)^2 - 2\right) \sin\left(\frac{\delta}{2}\right)^4. \end{cases} \quad (5.29)$$

$\boxed{\mathbf{g}^3 \cdot \mathbf{g}^3}$: The remaining class of contributions in the absence of interaction is depicted in fig.(5.9) and is given by

$$\begin{aligned} \mathbf{g}^3 \cdot \mathbf{g}^3 &= \frac{g^6}{N^2} P \oint_{C_1 C_2} \prod_{i=1}^3 dt_i ds_{i+3} \langle \text{Tr}[\mathbf{A}(t_1) \mathbf{A}(t_2) \mathbf{A}(t_3)] \text{Tr}[\mathbf{A}(s_4) \mathbf{A}(s_5) \mathbf{A}(s_6)] \rangle_0 + \\ &+ (C_1 \leftrightarrow C_2). \end{aligned} \quad (5.30)$$

For this family of graphs it is convenient to compute separately the three different contributions. The first one is similar to the diagrams considered in the previous

Figure 5.9: The three diagrams $g^3 \cdot g^3$.

cases. The sum of \mathbf{D}_2 and \mathbf{D}_3 in fig.(5.9) can be instead separated into the so-called *abelian* and *maximally non-abelian* part. To begin with, let us consider the diagram \mathbf{D}_1 which is given by

$$\mathbf{D}_1 = \frac{g^6 N}{32} \text{P} \oint_{C_1 C_2} \prod_{i=1}^3 dt_i ds_{i+3} \Delta_{12}(t_1, s_4) \Delta_{11}(t_2, t_3) \Delta_{22}(s_5, s_6) + (C_1 \leftrightarrow C_2). \quad (5.31)$$

Its evaluation is straightforward and one finds

$$\mathbf{D}_1 = \frac{g^6 N}{4A^6} (A_2 + A_3)(A_1 + A_3) A_1^2 A_2^2 = \begin{cases} \text{SYMMETRIC:} & \frac{g^6 N}{4} \cos\left(\frac{\delta}{2}\right)^4 \sin\left(\frac{\delta}{2}\right)^8 \\ \text{ASYMMETRIC:} & \frac{g^6 N}{128} \sin(\delta)^2 \sin\left(\frac{\delta}{2}\right)^2. \end{cases} \quad (5.32)$$

We come now to examine the *abelian* part, namely the part which is separately symmetric in (t_1, t_2, t_3) and (s_4, s_5, s_6) . We can exploit this symmetry to eliminate the path-ordering in the integral and to write

$$\mathbf{Ab} = \frac{g^6}{48N} \oint_{C_1 C_2} \prod_{i=1}^3 dt_i ds_{i+3} \Delta_{12}(t_1, s_4) \Delta_{12}(t_2, s_5) \Delta_{12}(t_3, s_6). \quad (5.33)$$

This integral is simply the cube of the single-exchange diagram and its value is

$$\mathbf{Ab} = \frac{g^6 N}{6A^6 N^2} A_1^3 A_2^3 = \begin{cases} \text{SYMMETRIC:} & \frac{g^6}{6N} \sin\left(\frac{\delta}{2}\right)^{12} \\ \text{ASYMMETRIC:} & \frac{g^6}{48N} \sin\left(\frac{\delta}{2}\right)^6. \end{cases} \quad (5.34)$$

Finally, we have to compute the *maximally non-abelian* part, whose expression is given by

$$\begin{aligned} \mathbf{NAb} = & \frac{g^6(N^3 - N)}{4N^2} \text{P} \oint_{C_1 C_2} \prod_{i=1}^3 dt_i ds_{i+3} \left[\Delta_{12}(t_1, s_4) \Delta_{12}(t_2, s_6) \Delta_{12}(t_3, s_5) + \right. \\ & \left. + \Delta_{12}(t_1, s_5) \Delta_{12}(t_2, s_4) \Delta_{12}(t_3, s_6) + \Delta_{12}(t_1, s_6) \Delta_{12}(t_2, s_5) \Delta_{12}(t_3, s_4) \right]. \end{aligned} \quad (5.35)$$

For two generic latitudes, we can perform five of the six integrations finding

$$\begin{aligned} \mathbf{NAb} = & \frac{g^6(N^3 - N)}{N^2} \left[\frac{J}{32\pi^2} \int_0^{2\pi} \frac{d\sigma \sigma^2 (\cos \theta_1 \cos \theta_2 - 1) (\cos \theta_2 - \cos \theta_1)^3}{\cos^2 \frac{\sigma}{2} (\cos \theta_2 - \cos \theta_1)^2 + (\cos \theta_1 \cos \theta_2 - 1)^2 \sin^2 \frac{\sigma}{2}} - \right. \\ & \left. - \frac{\pi J}{12} (\cos \theta_2 \cos \theta_1 - 1) (-2 \cos \theta_1 + \cos \theta_2 (\cos \theta_1 + 2) - 1) + 2\pi^3 J^3 \right] \equiv \\ & \equiv \frac{g^6(N^3 - N)}{N^2} \text{NAB}[\theta_1, \theta_2], \end{aligned} \quad (5.36)$$

where the constant J is defined by¹

$$J = \int_0^{2\pi} ds \Delta_{12}(t, s). \quad (5.37)$$

Actually we could also perform the last integration in terms of $\text{Li}_2(z)$, but for the subsequent numerical analysis this integral representation is more useful.

Let us collect the above results in a compact form. Apart from the *maximally non-abelian* contribution, all the other ladder graphs can be summed to give

$$\text{LAD}^{\text{SYM/ASYM}}[\delta] = g^6 N \text{LAD}_N^{\text{SYM/ASYM}}[\delta] + \frac{g^6}{N} \text{LAD}_{1/N}^{\text{SYM/ASYM}}[\delta], \quad (5.38)$$

where

$$\text{LAD}_N^{\text{SYM}}[\delta] = \frac{5}{12} \cos\left(\frac{\delta}{2}\right)^4 \sin\left(\frac{\delta}{2}\right)^8 + \frac{1}{3} \cos\left(\frac{\delta}{2}\right)^2 \sin\left(\frac{\delta}{2}\right)^{10}, \quad (5.39a)$$

$$\text{LAD}_N^{\text{ASYM}}[\delta] = \frac{1}{3072} \sin^2\left(\frac{\delta}{2}\right) (47 - 20 \cos(\delta) - 24 \cos(2\delta) + 4 \cos(3\delta) + \cos(4\delta)), \quad (5.39b)$$

$$\text{LAD}_{1/N}^{\text{SYM}}[\delta] = \frac{1}{12} \cos\left(\frac{\delta}{2}\right)^4 \sin\left(\frac{\delta}{2}\right)^8 + \frac{1}{6} \cos\left(\frac{\delta}{2}\right)^2 \sin\left(\frac{\delta}{2}\right)^{10} + \frac{1}{6} \sin\left(\frac{\delta}{2}\right)^{12}, \quad (5.39c)$$

$$\text{LAD}_{1/N}^{\text{ASYM}}[\delta] = \frac{1}{6144} \sin^2\left(\frac{\delta}{2}\right) (83 - 84 \cos(\delta) + 4 \cos(2\delta) + 4 \cos(3\delta) + \cos(4\delta)). \quad (5.39d)$$

¹The result does not depend on t since $\Delta_{12}(t, s) = \Delta_{12}(t - s)$.

The remaining *maximally non-abelian* part $\text{NAB}[\theta_1, \theta_2]$ can be evaluated numerically with high precision starting from expression (5.36), with irrelevant numerical error.

g^6 Interaction diagrams We now consider all the diagrams at order g^6 containing one or more interaction vertices. They could be divided into three different classes which are separately finite [H-DIAGRAM, X-DIAGRAM and IY-DIAGRAM].

H-DIAGRAM The H -diagram is the one depicted in fig.(5.10). Using the usual parametrization (5.15) we can write the contribution from this diagram as follows

$$\begin{aligned} \mathbf{H} = -\frac{\lambda^3}{8N^2} \int d^4w \left[P^M(x_1, y_1, w) \square_w^{\mathbf{A}_1} P^M(x_2, y_2, w) + P^M(x_1, y_1, w) \square_w^{\mathbf{B}_1} Q^M(x_2, y_2, w) + \right. \\ \left. + Q^M(x_1, y_1, w) \square_w^{\mathbf{B}_2} P^M(x_2, y_2, w) + Q^M(x_1, y_1, w) \square_w^{\mathbf{A}_2} Q^M(x_2, y_2, w) \right], \end{aligned} \quad (5.40)$$

where

$$P^M(x_i, y_i, w) = \int_0^{2\pi} d\tau_i d\sigma_i [2\dot{y}_i^M (\dot{x}_i \cdot \partial_{y_i} \mathcal{I}_i(x_i, y_i, w)) - 2\dot{x}_i^M (\dot{y}_i \cdot \partial_{x_i} \mathcal{I}_i(x_i, y_i, w))] \quad (5.41)$$

and

$$Q^M(x_i, y_i, w) = \int_0^{2\pi} d\tau_i d\sigma_i (\dot{x}_i \circ \dot{y}_i) (\partial_{x_i^M} \mathcal{I}_i(x_i, y_i, w) - \partial_{y_i^M} \mathcal{I}_i(x_i, y_i, w)). \quad (5.42)$$

In eqs.(5.40), (5.41) and (5.42) the index M is a ten-dimensional label running from 1 to 10. We have also defined $x^M \equiv (x^\mu, i\Theta^I|\dot{x}|)$, $\partial_M \equiv (\partial_\mu, 0)$ and indicated with \circ the ten-dimensional scalar product while with \cdot the four dimensional one. The spatial components P^μ of P^M satisfy the following two simple identities: $z_\mu P^\mu = \partial_\mu P^\mu = 0$, as can easily be checked by direct computation. Moreover, for two latitudes parallel to the plane (2,3), P^1 and P^4 trivially vanish. Since P^μ is a just a function of z^μ , all these properties are consistent if and only if $P^\mu = 0$. This result simplifies dramatically the computation for the correlator of two latitudes: in fact the contributions \mathbf{B}_1 and \mathbf{B}_2 in (5.40) are identically zero. Recall, in fact, that Q^M is different from zero (by construction) only when M is spatial. Thus we are just left with \mathbf{A}_1 and \mathbf{A}_2 to be computed.

To begin we first compute \mathbf{A}_2 . It is convenient to rewrite this contribution as follows

$$\mathbf{A}_2 = \frac{\lambda^3}{8N^2} \int_0^{2\pi} d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 (\dot{x}_1 \circ \dot{y}_1) (\dot{x}_2 \circ \dot{y}_2) (\partial_{x_1} - \partial_{y_1}) \cdot (\partial_{x_2} - \partial_{y_2}) \mathcal{H}(x_1, y_1; x_2, y_2), \quad (5.43)$$

where

$$\mathcal{H}(x_1, y_1; x_2, y_2) = \frac{1}{(2\pi)^{10}} \int \frac{d^4 z d^4 w}{(x_1 - z)^2 (y_1 - z)^2 (z - w)^2 (x_2 - w)^2 (y_2 - w)^2}. \quad (5.44)$$

The action of $(\partial_{x_1} - \partial_{y_1}) \cdot (\partial_{x_2} - \partial_{y_2})$ on $\mathcal{H}(x_1, y_1; x_2, y_2)$ can then be evaluated with the identity (A.7) given in [97]. One finds

$$\begin{aligned} (\partial_{x_1} - \partial_{y_1}) \cdot (\partial_{x_2} - \partial_{y_2}) \mathcal{H}(x_1, y_1; x_2, y_2) = \\ = \frac{1}{(x_1 - y_1)^2 (x_2 - y_2)^2} \left[\mathcal{I}^{(4)}(x_1, y_1, x_2, y_2) ((x_1 - x_2)^2 (y_1 - y_2)^2 - (x_1 - y_2)^2 (x_2 - y_1)^2) + \right. \\ \left. + \frac{1}{(2\pi)^2} (Y(x_1, x_2, y_2) - Y(y_1, x_2, y_2) + Y(x_2, x_1, y_1) - Y(y_2, x_1, y_1)) \right], \end{aligned} \quad (5.45)$$

where $Y(x_1, x_2, x_3) \equiv \mathcal{I}_1(x_1, x_2, x_3)[(x_1 - x_3)^2 - (x_1 - x_2)^2]$. Both in the SYMMETRIC and ASYMMETRIC case the integration over the circuits can now be carried numerically and one determines the color-stripped contribution $\mathbf{A2}[\theta_1, \theta_2]$ defined by

$$\mathbf{A2} = \frac{g^6(N^3 - N)}{N^2} \mathbf{A2}[\theta_1, \theta_2]. \quad (5.46)$$

Next we consider the evaluation of the $\mathbf{A1}$ contribution. We shall follow a different path in our analysis, namely we shall first perform the integration over the circuit analytically and then we perform numerically the integration over the position of the vertices. The first step is to study the function $P^M(w)$. The only non-vanishing components are $M = 4, 5$

$$P^4(w) = -2i \sin \theta_1 \sin \theta_2 (\cos \theta_1 - \cos \theta_2) (I_s(\theta_2) \partial_{w_0} I_c(\theta_1) - I_c(\theta_2) \partial_{w_0} I_s(\theta_1)), \quad (5.47a)$$

$$P^5(w) = -2i \sin \theta_1 \sin \theta_2 (\cos \theta_1 - \cos \theta_2) (I_s(\theta_2) \partial_{w_1} I_c(\theta_1) - I_c(\theta_2) \partial_{w_1} I_s(\theta_1)). \quad (5.47b)$$

where the function $I_c(\delta)$ and $I_s(\delta)$ are given in *appendix C*. In summary, we have to evaluate

$$\mathbf{A1} = \frac{g^6 N(N^2 - 1)}{8N^2} \int d^4 w d^4 z \frac{P^4(w) P^4(z) + P^5(w) P^5(z)}{(w - z)^2}. \quad (5.48)$$

Two of the eight integrations can be performed analytically (we do not present the cumbersome result): if we set

$$\mathbf{A1} = \frac{g^6 N(N^2 - 1)}{8N^2} \mathbf{A1}[\theta_1, \theta_2], \quad (5.49)$$

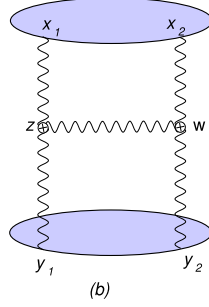


Figure 5.10: H-diagram

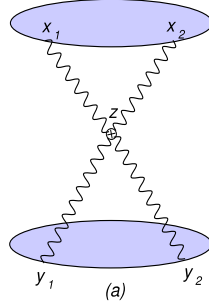


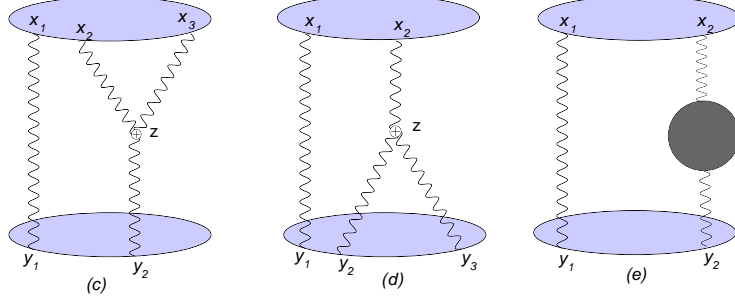
Figure 5.11: X-diagram

the remaining six integrals defining the quantity $\mathbf{A1}[\theta_1, \theta_2]$ can be computed numerically. This step is the most delicate one and the most unstable from the point of view of the convergence of the numerical integration.

X-DIAGRAM The second diagram to be considered is the so-called X-DIAGRAM (see fig. 5.11). Its expression is quite compact and it is given by

$$\begin{aligned} \mathbf{X} = & \frac{g^6 N(N^2 - 1)}{8(4\pi^2)^4} \int_0^{2\pi} dt_1 dt_2 ds_1 ds_2 \times \\ & \times \int d^4 w \frac{(x_1 \circ y_2)(x_2 \circ y_1) - (x_1 \circ x_2)(y_2 \circ y_1)}{(x_1 - w)^2 (x_2 - w)^2 (y_1 - w)^2 (y_2 - w)^2}. \end{aligned} \quad (5.50)$$

For the numerical evaluation the most convenient thing to do is to perform, first, the integration over the contours. Evaluating the integrals over the two circuits, for two generic latitudes we obtain the following expression in terms of $I[\delta]$, $I_c[\delta]$ and $I_s[\delta]$ defined in *appendix C*

Figure 5.12: The two “*IY*-diagrams” and the self-energy correction.

$$\begin{aligned} \mathbf{X} = & \frac{g^6 N(N^2 - 1)}{8(4\pi^2)^4 N^2} \int d^4 w \sin^4 \theta_1 \sin^4 \theta_2 [(I(\theta_1)^2 - I_c(\theta_1)^2 - I_s(\theta_1)^2)(I_c(\theta_2)^2 + \\ & + I_s(\theta_2)^2 - I(\theta_2)^2)] + [\sin \theta_1 \sin \theta_2 (1 - \cos \theta_1 \cos \theta_2)(I_c(\theta_1)I_c(\theta_2) + \\ & + I_s(\theta_1)I_s(\theta_2)) - \sin^2 \theta_1 \sin^2 \theta_2 I(\theta_1)I(\theta_2)]^2. \end{aligned} \quad (5.51)$$

If we define the function $\mathbf{X}[\theta_1, \theta_2]$ as

$$\mathbf{X} = \frac{g^6 N(N^2 - 1)}{N^2} \mathbf{X}[\theta_1, \theta_2], \quad (5.52)$$

for our specific configurations we can proceed with the numerical integration without encountering particular problems.

IY-DIAGRAM The last term that we have to consider corresponds to the sum of the diagrams depicted in fig.(5.12). We have two contributions that we call respectively \mathbf{IY}_{up} [(c) in fig.(5.12)] and $\mathbf{IY}_{\text{down}}$ [(d) in fig.(5.12)], and a diagram which takes into account the one-loop correction to the effective propagator [(e) in fig.(5.12)]. We shall denote this third diagram by **Budiag**. To begin with, we focus our attention on \mathbf{IY}_{up} , whose expression is

$$\mathbf{IY}_{\text{up}} = \frac{\lambda^3 J}{8N^2} \int_0^{2\pi} d\tau_1 d\tau_2 d\tau_3 d\sigma_2 \varepsilon(\tau_1, \tau_2, \tau_3) \{ (\dot{x}_1 \circ \dot{y}_2) \dot{x}_2 \cdot (\partial_{y_2} - \partial_{x_1}) - (\dot{x}_1 \circ \dot{x}_2) \dot{y}_2 \cdot \partial_{x_2} \} \mathcal{I}_1(x_1, x_2, y_2), \quad (5.53)$$

and on $\mathbf{IY}_{\text{down}}$, which is obtained from \mathbf{IY}_{up} by exchanging the roles of σ and τ (and therefore x_i and y_i). In its evaluation we encounter divergences at coincident points ($\tau_1 \rightarrow \tau_2$) in the integration over the upper circuit. This singularity though is compensated by the standard ultraviolet-divergence of the self-energy graph: half of diagram **Budiag** cancels the divergence for $\tau_1 \rightarrow \tau_2$, while the other half cancels

the same singularity in $\mathbf{IY}_{\text{down}}$ for $\sigma_1 \rightarrow \sigma_2$. To see this, we perform a trivial integration by parts and rewrite \mathbf{IY}_{up} in the following form

$$\begin{aligned} \mathbf{IY}_{\text{up}} = & \frac{\lambda^3 J}{8N^3} \left[\int_0^{2\pi} d\tau_1 d\tau_2 d\tau_3 d\sigma_2 \varepsilon(\tau_1, \tau_2, \tau_3) \{ (\dot{x}_1 \circ \dot{y}_2) 2\dot{x}_2 \cdot \partial_{y_2} - (\dot{x}_1 \circ \dot{x}_2) \dot{y}_2 \cdot \partial_{x_2} \} \mathcal{I}_1(x_1, x_2, y_2) + \right. \\ & \left. - 2 \int_0^{2\pi} d\tau_1 d\tau_2 d\sigma_2 (\dot{x}_1 \circ \dot{y}_2) \mathcal{I}_1(x_1, x_2, y_2) + \frac{1}{2} \int_0^{2\pi} d\tau_1 d\tau_3 d\sigma_2 (\dot{x}_1 \circ \dot{y}_2) \mathcal{I}_1(x_1, x_1, y_2) \right]. \end{aligned} \quad (5.54)$$

The singular part for coincident points is now singled out in the last term, which is proportional to the function $\mathcal{I}_1(x_1, x_1, y_2)$ (see *appendix C* for its properties). Since

$$\mathbf{Budiag} = -\frac{\lambda^3 J}{8N^2} \int_0^{2\pi} d\tau_1 d\tau_3 d\sigma_2 (\dot{x}_1 \circ \dot{y}_2) \mathcal{I}_1(x_1, x_1, y_2). \quad (5.55)$$

half of \mathbf{Budiag} exactly cancels the singularity present in \mathbf{Y}_{up} and we are left with

$$\begin{aligned} \mathbf{IY}_{\text{up}} = & \frac{\lambda^3 J}{8N^2} \left[\int_0^{2\pi} d\tau_1 d\tau_2 d\tau_3 d\sigma_2 \varepsilon(\tau_1, \tau_2, \tau_3) \{ (\dot{x}_1 \circ \dot{y}_2) 2\dot{x}_2 \cdot \partial_{y_2} - (\dot{x}_1 \circ \dot{x}_2) \dot{y}_2 \cdot \partial_{x_2} \} \mathcal{I}_1(x_1, x_2, y_2) - \right. \\ & \left. - 2 \int_0^{2\pi} d\tau_1 d\tau_2 d\sigma_2 (\dot{x}_1 \circ \dot{y}_2) \mathcal{I}_1(x_1, x_2, y_2) \right]. \end{aligned} \quad (5.56)$$

This expression does not exhibit any singularity at coincident points. The next step is to evaluate explicitly the integration over t_3 by means of the formula $\int_0^{2\pi} dt_3 \epsilon(t_1, t_2, t_3) = 2\pi \text{sign}(t_1 - t_2) - 2(t_1 - t_2)$. If we define the function $\mathbf{IY}[\theta_1, \theta_2]$ as follows

$$\mathbf{IY}_{\text{up}} + \mathbf{IY}_{\text{down}} = \frac{g^6(N^3 - N)}{N^2} \mathbf{IY}[\theta_1, \theta_2], \quad (5.57)$$

its value can be computed numerically both for the SYMMETRIC and for the ASYMMETRIC case.

5.4 Comparison with YM₂

We can now compare our numerical result with the analytic prediction given by two-dimensional Yang-Mills theory. Let us sum first all the contributions computed in the numerical analysis. The result is summarized for the SYMMETRIC CASE in fig.(5.13) while the ASYMMETRIC CASE is given in fig.(5.14).

The prediction in the present two cases can be derived easily from the general expression for the correlator in the zero instanton sector given in (5.8). For the SYMMETRIC CASE we find

$$\langle WW \rangle_{g^6} = \frac{g^6 N}{24} (5 + 4 \cos \delta + \cos \delta^2) \sin \left(\frac{\delta}{2} \right)^8 + \frac{g^6}{12N} \sin \left(\frac{\delta}{2} \right)^8, \quad (5.58)$$

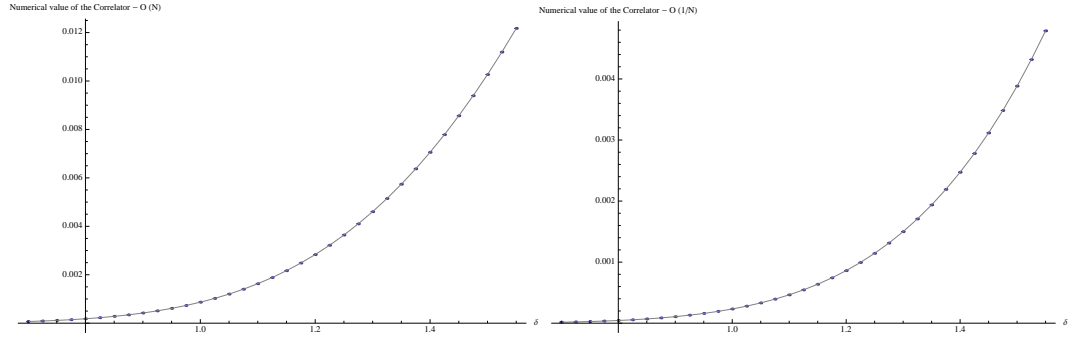


Figure 5.13: symmetric case: Leading ($g^6 N$) and sub-leading (g^6/N) contributions. The points are the results of the numerical analysis, while the light gray line is the YM_2 prediction.

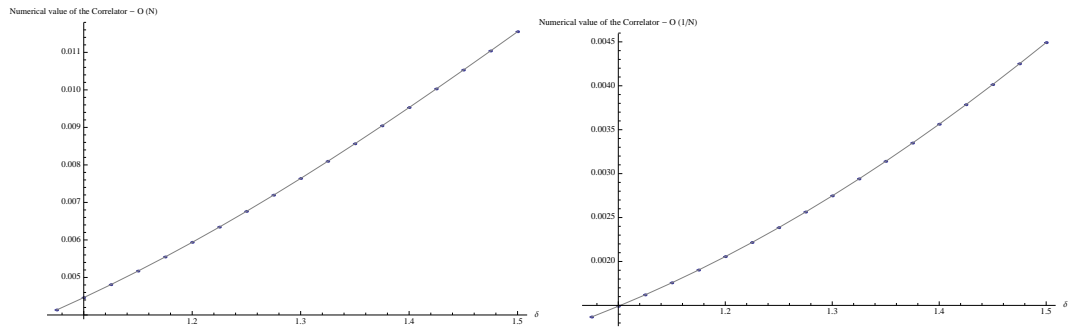


Figure 5.14: asymmetric case: Leading ($g^6 N$) and sub-leading (g^6) contributions. The points are the results of the numerical analysis, while the light gray line is the YM_2 prediction.

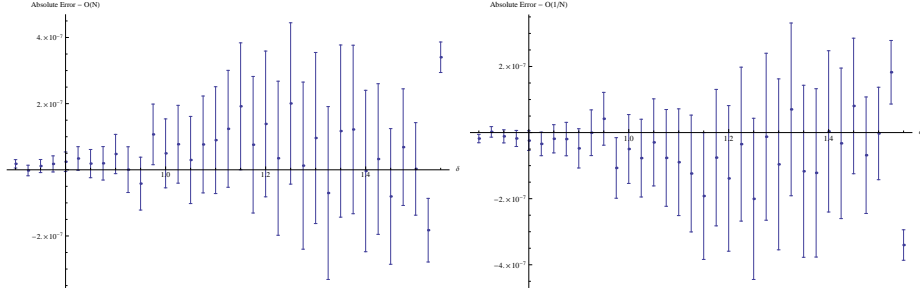


Figure 5.15: SYMMETRIC CASE: $\Delta_{\text{Cal-Pred}}$ at the leading ($g^6 N$) and sub-leading g^6/N contribution

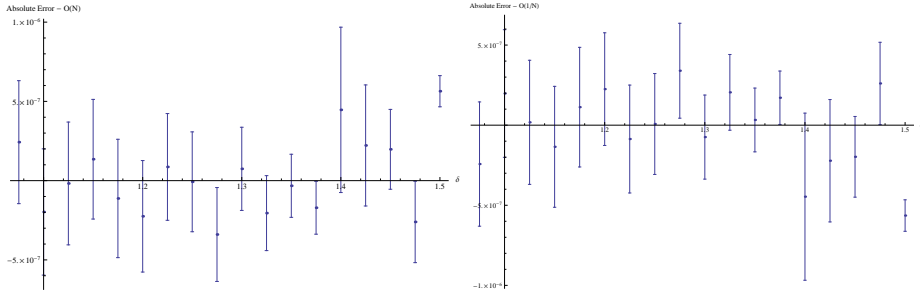


Figure 5.16: ASYMMETRIC CASE: $\Delta_{\text{Cal-Pred}}$ at the leading ($g^6 N$) and sub-leading g^6/N contribution

while for the ASYMMETRIC CASE we obtain

$$\begin{aligned} \langle WW \rangle_{g^6} = & \frac{g^6 N}{3072} \left(\sin^2 \left(\frac{\delta}{2} \right) (-36 \cos(\delta) - 20 \cos(2\delta) + 4 \cos(3\delta) + \cos(4\delta) + 59) \right) + \\ & \frac{g^6}{6144 N} \left(\sin^2 \left(\frac{\delta}{2} \right) (-52 \cos(\delta) - 4 \cos(2\delta) + 4 \cos(3\delta) + \cos(4\delta) + 59) \right). \end{aligned} \quad (5.59)$$

In order to compare the results presented in figs.(5.13) and (5.14) with the answer of matrix model, we compute the difference $\Delta_{\text{Cal-Pred}}$ between the calculated values and the predicted ones. The results of this analysis are plotted in figs.(5.4) and (5.4), where the bar denotes the estimated errors. We see that the difference from the central value 0 is quite small. It is easy to see that the average absolute error is of order $10^{-6} / 10^{-7}$, the relative error is of the order 10^{-4} at worst.

We also note that the error bars increase as the coincident limit is approached. This is a generic feature of the calculation as the integrands become increasingly singular in this limit. Conversely, the error is relatively small for small δ , however the precision required to reliably calculate certain integrals in this “shrinking” limit becomes prohibitive for δ less than about 0.7 (1.0), in the SYMMETRIC CASE

(ASYMMETRIC CASE), and this defines the lower bound of our chosen range. We thus conclude that the conjecture is verified with a relative error of order 10^{-4} in the range $0.7 \leq \delta \leq \pi/2$ for the SYMMETRIC CASE and in the range $1 \leq \delta \leq \pi/2$ for the ASYMMETRIC CASE.

5.5 Operator Product Expansion

In this subsection we will present another perturbative result, in particular, we will consider the limit in which one of the latitudes shrinks to a point at the pole of the sphere. This limit is motivated by the fact that the emerging structure can be usefully understood in terms of the OPE and its physical meaning is quite transparent.

The crucial observation is that, viewed from a comparably large distance, the unshrunk Wilson loop sees the shrunken loop as a collection of local operators [95]: the quantum behavior is encoded into Wilson coefficients and anomalous dimensions. The story was worked out in detail for two circular Wilson-Maldacena loops in [94]. Here, for the 1/4 BPS latitude, we will find that the relevant OPE is quite different, giving rise to novel operators which appear to have protected dimensions.

When analysing the OPE, we can in fact consider the general situation of loops with *arbitrary* contours on S^2 that are generically 1/8 BPS. As noticed in [60] the Wilson loop on S^2 can be written in terms of a new gauge connection

$$\mathcal{A}_i = A_i + i\epsilon_{ijk} x^j \frac{\Phi^k}{R}. \quad (5.60)$$

The OPE expansion will appear particularly simple using this generalized connection. The first step is to determine the classical expansion of our Wilson loops in terms of local gauge-invariant operators when the circuit is small. To achieve this goal we shall assume that the circuit can be written as follows

$$x^i(t) = x_0^i + r\hat{x}^i(t), \quad (5.61)$$

x_0 being the point about which the loop is shrinking and r a parameter that will control the limit. We expand the contour integral by exploiting the Fock-Schwinger gauge $(x - x_0)^i \mathcal{A}_i(x) = 0$, where the following formula holds in terms of the new gauge curvature \mathcal{F}_{ji}

$$\mathcal{A}_i(x) = \int_0^1 d\lambda \lambda (x - x_0)^j \mathcal{F}_{ji}(x_0 + \lambda(x - x_0)). \quad (5.62)$$

The leading order result is given by

$$\oint_C dt \mathcal{A}_i(x) \dot{x}^i = \frac{r^2}{2} \mathcal{F}_{ij}(x_0) \oint_C dt \hat{x}^i(t) \dot{\hat{x}}^j(t) + O(r^3) = \frac{r^2}{2} \epsilon^{ijk} \mathcal{F}_{ij}(x_0) n_k(x_0) + O(r^3), \quad (5.63)$$

$n_i(x_0)$ being a normal vector to S^2 at the point x_0 , depending on x_0 and the contour. The expansion could of course be extended to any given order in r , producing a series of local operators $\mathcal{O}_C^J(x)$ determined by the particular shape of the Wilson loop, the generalized connection \mathcal{A}_i itself depending on the contour. Because these operators should share the BPS properties of the associated Wilson loop, we obtain a practical realization of the proposal of [96]: in particular we could expect that their correlation functions, when restricted to the relevant S^2 , are somehow protected from quantum corrections.

In our specific example we take as our shrinking point the north pole, $x_0 = R(0, 0, 1)$, while $r = \sin \theta_0$ and $\hat{x}^i(t) = R(\cos t, \sin t, \tan \frac{\theta_0}{2})$. Due to the trace in the path-exponential the first non-vanishing contribution to the OPE is quadratic in the fields, and we get explicitly at leading order

$$W_0 = 1 + \frac{\pi^2 \sin^4 \theta_0}{2N} \mathcal{O}_{\mathcal{F}}(x_0) \quad (5.64)$$

where

$$\mathcal{O}_{\mathcal{F}}(x_0) = \text{Tr} \left[2R \Phi_3 - iR^2 F_{12} - R^2 (\partial_1 \Phi_1 + \partial_2 \Phi_2) \right]^2. \quad (5.65)$$

We note a peculiar feature that makes this OPE quite different from the usual circular Wilson-Maldacena case [94]: operators of classical dimension 2, 3, and 4 all couple with the same power of the parameter which sets the size of the shrinking latitude: the polar angle θ (in the standard case the power is the classical dimension itself). Indeed the overall scale R of the S^2 is just a place keeper. The conformality of $\mathcal{N} = 4$ SYM prevents it from playing any rôle, and it drops out of the calculation of any observable.

We notice that we can easily obtain the leading term of the two latitude correlator at order g^4 from the OPE (5.64), once we restore the canonical normalization for the fields. We just need to compute the correlation function

$$\langle \mathcal{O}_{\mathcal{F}}(x_0) \oint dt \left(\dot{x}_1^i A_i(x_1) - i\epsilon_{ijk} \dot{x}_1^j \dot{x}_1^k \Phi^i(x_1) \right) \rangle = i \frac{\cos(\theta_1) + 1}{4\pi}, \quad (5.66)$$

that enters in the Wick contraction. Taking the relevant color traces we get

$$\frac{\langle W_0 W_1 \rangle}{\langle W_0 \rangle \langle W_1 \rangle} - 1 = \frac{g^4}{8} \left(2 \sin^2 \frac{\theta_0}{2} \cos^2 \frac{\theta_1}{2} \right)^2 = \frac{g^4 r^4}{32} \cos^2 \frac{\theta_1}{2}. \quad (5.67)$$

Actually we can learn something more: the general expectation for the structure of the OPE of a shrinking Wilson loop is given by [94, 95, 98]

$$\frac{W}{\langle W \rangle} = 1 + \sum_J \xi_J(g^2) L^{\Delta_J} O_J(x) \quad (5.68)$$

where L is the size of the shrinking loop, and $O_J(x)$ is an operator of classical dimension J and quantum dimension $\Delta_J = J + g^2 \Delta_J^{(1)} + \dots$. The Wilson coefficients

$\xi_J(g^2)$ depend on the coupling constant g^2 . The curious structure of the latitude OPE is a reflection of the fact that the coefficients $\xi_J(g^2)$ which describe the coupling of the Wilson loop to a specific operator $O_J(x)$ are themselves functions of θ [107], and can be expanded as $\xi_J(g^2, \theta) = \sum_k \xi_J^{(k)}(g^2) \theta^k$ in the limit $\theta \rightarrow 0$. This provides us with the general structure for the OPE of W_0

$$\begin{aligned} \frac{W_0}{\langle W_0 \rangle} &= 1 + \sum_J \xi_J(g^2, \theta_0) \theta_0^{\Delta_J} O_J(x_0) = 1 + \sum_{J,k} \xi_J^{(k)}(g^2) \theta_0^{\Delta_J+k} O_J(x_0) = \\ &= 1 + \xi_2^{(2)} \theta_0^{\Delta_2+2} O_2(x_0) + \xi_3^{(1)} \theta_0^{\Delta_3+1} O_3(x_0) + \xi_4^{(0)} \theta_0^{\Delta_4} O_4(x_0) + \dots, \end{aligned} \quad (5.69)$$

where we have dropped the scale R (to restore it replace $O_J(x_0) \rightarrow R^{\Delta_J} O_J(x_0)$), and have noted the vanishing of $\xi_2^{(0,1)}$ and $\xi_3^{(0)}$ from the explicit expression of (5.65). The explicit form of $O_{2,3,4}(x_0)$ is simply obtained from $\mathcal{O}_{\mathcal{F}}(x_0)$. Actually there are multiple operators of the same classical dimension, so there is an extra suppressed index on the $\xi_J(g^2, \theta_0)$, Δ_J , and $O_J(x_0)$, which is implicitly summed over in (5.67). In the last line we are referring only to the operators appearing in (5.65) as these are the only ones present at leading order in θ_0 . We derive the following general relation in the shrinking limit

$$\frac{\langle W_1 W_0 \rangle}{\langle W_1 \rangle \langle W_0 \rangle} = 1 + \xi_2^{(2)} \theta_0^{\Delta_2+2} \langle W_1 O_2(x_0) \rangle + \xi_3^{(1)} \theta_0^{\Delta_3+1} \langle W_1 O_3(x_0) \rangle + \xi_4^{(0)} \theta_0^{\Delta_4} \langle W_1 O_4(x_0) \rangle + \dots \quad (5.70)$$

We notice that when expanded at small coupling the $\theta_0^{\Delta_J}$ terms generically produce logarithms $\theta_0^{\Delta_J} = \theta_0^J + g^2 \Delta_J^{(1)} \theta_0^J \log \theta_0 + \dots$ if quantum corrections modify the classical dimensions. The quantities $\xi_2^{(2)}$, $\xi_3^{(1)}$, and $\xi_4^{(0)}$ may easily be read-off in our case from (5.65). Since the operators appearing in the explicit expression are quadratic in the fields, one has that $\xi_2^{(2)}$, $\xi_3^{(1)}$, and $\xi_4^{(0)}$ lead as g^4 . We therefore generally expect terms of the form $g^6 \log \theta_0$ to show up in the perturbative expansion of the correlator at order g^6 , in the shrinking limit.

The presence of logarithmic corrections would be a signal that anomalous dimensions are playing a part, suggesting that the full interacting theory should be taken into account and localization techniques would not be sufficient in the exact computation. It would also rule out the relation with two-dimensional Yang-Mills that produces just polynomial dependence on θ at any order of perturbation theory, as we will see in section 5.2. In the following we show that, surprisingly, no such logarithmic terms appear at order g^6 , supporting the matrix model proposal. This indicates that the composite operator $\mathcal{O}(x)$, arising from the OPE of the BPS loops, should be protected, at least at the first non-trivial quantum order. In other words logarithmic divergences should be absent in the two-point function $\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle$, when $x_{1,2}$ belong to the relevant S^2 , in the same way as the operators defined in [96]. It is not difficult to show in fact that $\mathcal{O}(x)$ inherits the BPS properties of the

latitude loop, and a certain amount of supersymmetry is preserved by its correlators.

For this computation, we limit our attention to the gauge group $SU(N)$ and we can separate the diagrams into two classes: the ladder diagrams and the interaction diagrams. It is easy to realize that the ladder diagrams cannot generate any contribution of the form $r^k \log(r)$. They are actually analytic in the small r -limit. The contributions $r^k \log(r)$ are instead generated by the interactions diagrams in figs. (5.10)(5.11) and (5.12). The origin of this non analytic behavior can be traced back to the small distance singularities appearing in the integration over the position of the vertices. Thus in order to extract these logarithmic singularities, we have to first perform these integrations analytically, and only after that can we expand in powers of the radius. To illustrate the procedure let us start by considering the X -diagram. Its expression can be cast into the compact form

$$\mathbf{X} = \frac{\lambda^3}{8N^2} \int_0^{2\pi} d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 [(\dot{x}_1 \circ \dot{y}_2)(\dot{x}_2 \circ \dot{y}_1) - (\dot{x}_1 \circ \dot{x}_2)(\dot{y}_1 \circ \dot{y}_2)] \mathcal{I}^{(4)}(x_1, x_2, y_1, y_2), \quad (5.71)$$

where $(\dot{x} \circ \dot{y}) = \dot{x} \cdot \dot{y} - |\dot{x}||\dot{y}|\Theta_{\dot{x}} \cdot \Theta_{\dot{y}}$ with $|\dot{x}|\Theta_{\dot{x}}^I = M_I^i \epsilon_{irs} \dot{x}^r x^s$ and

$$\mathcal{I}^{(4)}(x_1, x_2, y_1, y_2) \equiv \frac{1}{(2\pi)^8} \int \frac{d^4 w}{(x_1 - w)^2 (x_2 - w)^2 (y_1 - w)^2 (y_2 - w)^2}. \quad (5.72)$$

Differently from the numerical analysis where first the integration over circuits has been performed, here we integrate over w (the position of the vertex). Then it is straightforward to extract the singular part when we shrink the latitude $\theta = \theta_1$ to the north-pole of the sphere S^2 . It is given by

$$\mathcal{I}^{(4) \text{ sing.}}(x_1, x_2, y_1, y_2) = -\frac{\log r}{128\pi^6} \times \int_0^1 \frac{d\alpha}{(1-\alpha)(y_1-x_2)^2(y_2-x_1)^2 - \alpha(1-\alpha)(x_1-x_2)^2(y_1-y_2)^2 + \alpha(y_1-x_1)^2(y_2-x_2)^2}, \quad (5.73)$$

where $r = \sin \theta_1$. Now the integration over the circuit can be easily performed and can be evaluated by Taylor-expanding in r . At leading order we find that

$$\mathbf{X}^{\text{sing}} = \frac{5r^4 \cos^4\left(\frac{\theta_2}{2}\right) \log(r)}{768\pi^2} + O(r^5). \quad (5.74)$$

Consider now the H -diagram in fig.(5.10). As we have seen in the previous section we are just left with \mathbf{A}_1 and \mathbf{A}_2 contribution.

Let us first compute first \mathbf{A}_2 . In eq.(5.43), when the first latitude ($\theta = \theta_1$) is shrunk to zero the logarithmically divergent terms can be generated by $\mathcal{I}^{(4)}$ and by the Y that depends both on x_1 and x_2 . Therefore we can write

$$\mathbf{A}_2^{\text{sing.}} = \frac{\lambda^3}{8N^2} \int_0^{2\pi} d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 \frac{(\dot{x}_1 \circ \dot{y}_1)(\dot{x}_2 \circ \dot{y}_2)}{(x_1 - y_1)^2 (x_2 - y_2)^2} \left[\mathcal{I}^{(4)\text{sing.}}(x_1, y_1, x_2, y_2) ((x_1 - x_2)^2 (y_1 - y_2)^2 - (x_1 - y_2)^2 (x_2 - y_1)^2) + \frac{1}{(2\pi)^2} (Y^{\text{sing.}}(x_1, x_2, y_2) + Y^{\text{sing.}}(x_2, x_1, y_1)) \right], \quad (5.75)$$

where we have defined

$$Y^{\text{sing.}}(x_1, x_2, y_2) \equiv \mathcal{I}_1^{\text{sing.}}(x_1, x_2, y_2) [(x_1 - y_2)^2 - (x_1 - x_2)^2]$$

and

$$Y^{\text{sing.}}(x_2, x_1, y_2) \equiv \mathcal{I}_1^{\text{sing.}}(x_1, x_2, y_2) [(x_2 - y_2)^2 - (x_2 - x_1)^2].$$

(the expression for $\mathcal{I}_1^{\text{sing.}}$ is given in *appendix (C)*). The integration over the circuits can then be easily performed with the help of Mathematica if we first expand the integrand of (5.75) in powers of r . At leading order we find

$$\mathbf{A}_2^{\text{sing.}} = -\frac{7r^4 \cos^4\left(\frac{\theta_2}{2}\right) \log(r)}{1536\pi^2} + O(r^5). \quad (5.76)$$

Now we have to compute the contribution \mathbf{A}_1 . To extract the divergent part we follow a different approach respect to the perturbative g^6 computation of the previous sections. The first step is to add two total derivatives to the integrand of P^M

$$P^M(x_1, y_1, w) = \int_0^{2\pi} d\tau_1 \int_0^{2\pi} d\sigma_1 \left[2\dot{y}_1^M \underbrace{(\dot{x}_1 \cdot \partial_{y_1} \mathcal{I}_1(y_1 - w, x_1 - w) - \dot{x}_1 \cdot \partial_{x_1} \mathcal{I}_2(y_1 - w, x_1 - w))}_{K_1} - 2\dot{x}_1^M \underbrace{(y_1 \cdot \partial_{x_1} \mathcal{I}_1(x_1 - w, y_1 - w) - y_1 \cdot \partial_{y_1} \mathcal{I}_2(x_1 - w, y_1 - w))}_{K_2} \right]. \quad (5.77)$$

These two new terms obviously yield a vanishing result when the integration runs along the circuits. Since the following identity for \mathcal{I}_1 and \mathcal{I}_2 holds (see *appendix C*)

$$\frac{\partial}{\partial x^\mu} \mathcal{I}_1(x, y) - \frac{\partial}{\partial y^\mu} \mathcal{I}_2(x, y) = -\frac{1}{32\pi^4} \frac{x^\mu}{x^2} \frac{\log\left(\frac{(x-y)^2}{y^2}\right)}{[(x-y)^2 - y^2]}, \quad (5.78)$$

the combination K_1 appearing in P^M can be rearranged in the following compact

form

$$K_1 = -\frac{1}{64\pi^4(y_1-w)^2} \frac{d}{d\tau_1} \left[\text{Li}_2 \left(1 - \frac{(x_1-y_1)^2}{(x_1-w)^2} \right) + \frac{1}{2} \left(\log \left[\frac{(x_1-w)^2}{(x_2-y_2)^2} \right] \right)^2 \right] + \frac{1}{32\pi^4} \frac{(x_1-w) \cdot \dot{x}_1}{(x_1-w)^2(y_1-w)^2} \log \left(\frac{(x_1-y_1)^2}{(x_2-y_2)^2} \right). \quad (5.79)$$

The combination K_2 can be also recast into the same form. The only difference from (5.79) is that the roles of x_1 and y_1 , and of τ_1 and σ_1 , are exchanged. The terms in K_1 and K_2 that are total derivatives with respect to τ_1 and σ_1 can be dropped since they yield a vanishing contribution to P^M , and we are left with the compact expression

$$P^M(x_1, y_1, w) = \frac{1}{16\pi^4} \int_0^{2\pi} d\tau_1 d\sigma_1 \frac{\dot{y}_1^M (x_1-w) \cdot \dot{x}_1 - \dot{x}_1^M (y_1-w) \cdot \dot{y}_1}{(x_1-w)^2(y_1-w)^2} \log \left(\frac{(x_1-y_1)^2}{(x_2-y_2)^2} \right). \quad (5.80)$$

Then, if we take into account that

$$-\square_w P^M(x_2, y_2, w) = \int_0^{2\pi} d\tau_1 d\sigma_1 [2\dot{y}_2^M \dot{x}_2 \cdot \partial_{y_2} - 2\dot{x}_2^M \dot{y}_2 \cdot \partial_{x_2}] \frac{1}{(2\pi)^4} \frac{1}{(x_2-w)^2(y_2-w)^2}, \quad (5.81)$$

we can rewrite the \mathbf{A}_1 contribution in the following form

$$\mathbf{A}_1 = \frac{\lambda^3}{4N^2} \int_0^{2\pi} d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 \log \left(\frac{(x_1-y_1)^2}{(x_2-y_2)^2} \right) \left[[(\dot{y}_1 \circ \dot{y}_2) \dot{x}_2 \cdot \partial_{y_2} - (\dot{y}_1 \circ \dot{x}_2) \dot{y}_2 \cdot \partial_{x_2}] \times \right. \\ \left. \times \dot{x}_1 \cdot S(x_1, x_2, y_1, y_2) - [(\dot{x}_1 \circ \dot{y}_2) \dot{x}_2 \cdot \partial_{y_2} - (\dot{x}_1 \circ \dot{x}_2) \dot{y}_2 \cdot \partial_{x_2}] \dot{y}_1 \cdot S(x_1, x_2, y_1, y_2) \right] \quad (5.82)$$

where

$$S^\mu(x_1, x_2, y_1, y_2) \equiv -\frac{1}{(4\pi^2)^4} \int d^4 w \frac{w^\mu}{(x_1-w)^2(y_1-w)^2(x_2-w)^2(y_2-w)^2}. \quad (5.83)$$

The nice feature of (5.82) is the disappearance of one of the integrations over the position of the vertices. Although this result simplifies the procedure for extracting the logarithmic terms appearing in the limit $\theta_1 \rightarrow 0$, the computation is still a little bit cumbersome and some of the details are given in appendix A of [78]. Here we shall only give the final result after the integration over the circuits. At the leading order in $r(\equiv \sin \theta_1)$, we find

$$\mathbf{A}_1 = \frac{r^4 \cos^4 \left(\frac{\theta_2}{2} \right) \log(r)}{512\pi^2} + O(r^5) \quad (5.84)$$

The final set of diagrams to consider are the so called IY-diagram. First we note that the logarithmic part arising when we shrink the upper circle to a point is then obtained by replacing \mathcal{I}_1 in the expression (5.56) with the $\mathcal{I}_1^{\text{sing.}}$ found in *appendix C*. Next we Taylor-expand in r and integrate over the circuits. At leading order in r we find

$$\mathbf{IY}_{\text{up}}^{\text{sing.}} = -\frac{r^4 \cos^4\left(\frac{\theta_2}{2}\right) \log(r)}{256\pi^2} + O(r^5) \quad (5.85)$$

Let us now sum all the different contributions at leading order in r

$$\mathbf{X}^{\text{sing.}} + \mathbf{IY}_{\text{up}}^{\text{sing.}} + \mathbf{A}_1^{\text{sing.}} + \mathbf{A}_2^{\text{sing.}} = \frac{r^4 \cos^4\left(\frac{\theta_2}{2}\right) \log(r)}{\pi^2} \left(\frac{5}{768} - \frac{1}{256} + \frac{1}{512} - \frac{7}{1536} \right) = 0! \quad (5.86)$$

Namely, we have verified that the logarithmic singularities cancel at the first non trivial order. This implies that the effective anomalous dimension of the operator $\mathcal{O}_{\mathcal{F}}$ vanishes at one-loop, supporting the idea that this operator is actually protected.

5.6 Correlator at strong coupling from supergravity

At strong coupling, the AdS/CFT correspondence may be used to compute the correlator between two Wilson loops in $\mathcal{N} = 4$ SYM [95] and thus to test our conjecture that relates them to the analogous observable in the bosonic YM₂. Practically, by taking the limit in which the separation of the two Wilson loops is much larger than their sizes, and working at large N , the correlator is computed by calculating the exchange of light supergravity fluctuations between the Wilson loop worldsheets. At infinite separation, only the lightest fluctuation modes need be considered; the subleading contributions stemming from the relaxation of this limit are given by the exchange of heavier modes.

In the following we show that only one of the contributions to the correlator from a class of modes (dual to the $\mathcal{N} = 4$ SYM chiral primary operators $\text{Tr}(\Phi_3 + i\Phi_4)^J$) which includes a representative of the lightest modes (i.e. for $J = 2$) matched the YM₂ result. Beyond this, a remarkable pattern of matching between contributions from the modes dual to $\text{Tr}(\Phi_3 + i\Phi_4)^J$ for general J , and corresponding terms in the YM₂ expression for the correlator was uncovered. This remarkable pattern of matching terms has since been corroborated using the techniques of localization [101], where it was shown that the localization conditions equate the superprotected operator appearing in the Wilson loop's OPE expansion to precisely the chiral primary operator referred to above.

Beyond this pattern of matching terms, at respectively subleading orders in the large separation limit, the contributions of the aforementioned dual chiral primary modes also include terms *absent* from the YM₂ result. One would expect these terms to be removed, i.e. canceled, by the inclusion of the other supergravity

modes which are respectively heavier, order-by-order, compared to the dual chiral primaries.

In a correlator calculation we are instructed to sum over the exchange of all possible modes. Let us concentrate on the bottom of the spectrum. In addition to the mode dual to $\text{Tr}(\Phi_3 + i\Phi_4)^2$, one must also include the mode dual to the conjugate operator $\text{Tr}(\Phi_3 - i\Phi_4)^2$, and the orthogonal operators $\text{Tr}(\Phi_3^2 + \Phi_4^2 - \Phi_5^2 - \Phi_6^2)$ and $\text{Tr}(3(\Phi_1^2 + \Phi_2^2) - 1)^2$. These correspond in the supergravity picture to various S^5 spherical harmonics of weight 2. These extra modes contribute at the leading order in the large-separation limit, and in order not to spoil the agreement with YM_2 should be cancelled by yet other modes.

It happens that there are two types of further supergravity fluctuations around $AdS_5 \times S^5$ which could potentially do the job. These are the leading fluctuations of the NS-NS B-field with legs in the AdS_5 and S^5 directions respectively [100][102]. They are dual to the following gauge theory operators (see *appendix A* of [103]) $\psi_A \psi_B \rightarrow$ B-field on S^5 and $\bar{\psi}^A \sigma_{\mu\nu} \bar{\psi}^B + 2i\Phi^{AB} F_{\mu\nu}^+ \rightarrow$ B-field on AdS_5 . The coupling of these operators has been discussed previously in the context of the 1/2 BPS circular Wilson loop [94][104]. We will find that they provide leading contributions of the right order and sign, but fail to cancel the offending chiral primary contributions due to mismatched coefficients. By going higher in the supergravity spectrum, we have verified that the next heaviest modes all contribute beyond the leading order and are thus powerless to save the agreement with YM_2 .

The interpretation of the disagreement is not clear. The strong coupling limit here could be subtle, since we are considering Wilson loops in the limit in which they become the supersymmetric circles of Zarembo [57], where the rescaled coupling in the matrix model approaches zero [105]. Of course, there is also the possibility that the YM_2 -DGRT Wilson loop equivalence needs to be adjusted at strong coupling, perhaps through the effects of the undetermined 1-loop determinant appearing in the localization formulae [77]. Of course there may also be a subtlety with the supergravity calculations themselves.

Preliminaries

We briefly review the fundamental string solution corresponding to the latitude DGRT Wilson loop illustrated in chapter three. We write the metric of $AdS_5 \times S^5$ as

$$ds^2 = \left(\frac{dy^2 + dr^2 + r^2 d\phi^2 + dx^2 + dz^2}{y^2} + \cos^2 \vartheta d\Omega_3^2 + d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \quad (5.87)$$

where the angle $\vartheta \in [0, \pi/2]$. The worldsheet coordinates are $\sigma \in [0, \infty)$ and $\tau \in [0, 2\pi)$. The embedding functions are $z = d\Omega_3 = 0$ and

²Other possible modes with $J = 2$ do not couple to the Wilson loop, see appendix D

$$y = \sin \theta_i \tanh \sigma, \quad r = \frac{\sin \theta_i}{\cosh \sigma}, \quad \phi = \tau, \quad x = \cos \theta_i, \quad \sin \vartheta = \frac{1}{\cosh(\sigma + \sigma_i)}, \quad \varphi = \tau - \pi, \quad (5.88)$$

where θ_i is the position of the latitude on the sphere, i.e. its radius. Note that the latitude's path on the internal-space sphere is also a latitude, albeit at

$$\vartheta_i = \frac{\pi}{2} - \theta_i \quad (5.89)$$

and so

$$\sin \vartheta_i = \frac{1}{\cosh \sigma_i} = \cos \theta_i. \quad (5.90)$$

We would like to compute the correlator between two such latitudes at polar angles θ_0 and θ_1 , in the limit $\theta_0 \rightarrow 0$, $\theta_1 \rightarrow \pi$. In the rest of the document we take $\theta_1 \rightarrow \pi - \theta_1$, so that small θ_1 indicates a latitude close to the south pole of the S^2 .

Dual chiral primaries

The supergravity modes that we are interested in are fluctuations of the RR 5-form as well as the spacetime metric. They are by now very well known, and details can be found in [95][100][106][108][109]. The fluctuations are

$$\begin{aligned} \delta g_{\mu\nu} &= \left[-\frac{6J}{5} g_{\mu\nu} + \frac{4}{J+1} D_{(\mu} D_{\nu)} \right] s^J(X) Y_J(\Omega), \\ \delta g_{\alpha\beta} &= 2J g_{\alpha\beta} s^J(X) Y_J(\Omega), \end{aligned} \quad (5.91)$$

where μ, ν are AdS_5 and α, β are S^5 indices. The symbol X indicates coordinates on AdS^5 and Ω coordinates on the S^5 . The $D_{(\mu} D_{\nu)}$ represents the traceless symmetric double covariant derivative. The $Y_J(\Omega)$ are the spherical harmonics on the five-sphere, while $s^J(X)$ have arbitrary profile and represent a scalar field propagating on AdS_5 space with mass squared $= J(J-4)$, where J labels the representation of $SO(6)$ and must be an integer greater than or equal to 2.

The bulk-to-bulk propagator for s^J is given in [95], with normalization from [106]. It is expressed in terms of a hypergeometric function

$$\begin{aligned} P(X, \bar{X}) &= \frac{\alpha_0}{B_J} W^J {}_2F_1(J, J-3/2, 2J-3; -4W), \\ W &= \frac{y\bar{y}}{(y-\bar{y})^2 + (x-\bar{x})^2 + (z-\bar{z})^2 + r^2 + \bar{r}^2 - 2r\bar{r}\cos(\phi-\bar{\phi})}, \end{aligned} \quad (5.92)$$

where,

$$\alpha_0 = \frac{J-1}{2\pi^2}, \quad B_J = \frac{2^{3-J} N^2 J(J-1)}{\pi^2 (J+1)^2}. \quad (5.93)$$

Then we construct the traceless symmetric double covariant derivative,

$$D_{(\mu}D_{\nu)} \equiv \frac{1}{2}(D_{\mu}D_{\nu} + D_{\nu}D_{\mu}) - \frac{1}{5}g_{\mu\nu}g^{\rho\sigma}D_{\rho\sigma}, \quad (5.94)$$

the details of which are given in *appendix E*. Using a 10-d index $M = (\mu, \alpha)$, we can express the metric fluctuations as $\delta g_{MN} = \delta \tilde{g}_{MN} s^J Y_J$, where

$$\begin{aligned} \delta \tilde{g}_{yy} &= \frac{4}{J+1} \left[\Phi_y^2 + \frac{1}{y} \Phi_y \right] - \frac{2J(J-1)}{y^2(J+1)}, \quad \delta \tilde{g}_{rr} = \frac{4}{J+1} \left[\Phi_r^2 - \frac{1}{y} \Phi_y \right] - \frac{2J(J-1)}{y^2(J+1)}, \\ \delta \tilde{g}_{yr} &= \frac{4}{J+1} \left[\Phi_y \Phi_r + \frac{1}{y} \Phi_r \right], \quad \delta \tilde{g}_{\phi\phi} = \frac{4}{J+1} \left[\Phi_{\phi}^2 - \frac{r^2}{y} \Phi_y + r \Phi_r \right] - r^2 \frac{2J(J-1)}{y^2(J+1)}, \\ \delta \tilde{g}_{\vartheta\vartheta} &= 2J, \quad \delta \tilde{g}_{\varphi\varphi} = 2J \sin^2 \vartheta, \end{aligned} \quad (5.95)$$

and where we have used the fact that $D^2 s^J = J(J-4)s^J$. We may now assemble the expression for the correlator

$$\frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} = \left(\frac{\sqrt{\lambda}}{4\pi} \right)^2 \int_{\Sigma} \int_{\bar{\Sigma}} \partial_a X^M \partial^a X^N \delta g_{MN} P(X, \bar{X}) \delta \bar{g}_{\bar{M}\bar{N}} \partial_{\bar{a}} X^{\bar{M}} \partial^{\bar{a}} X^{\bar{N}}.$$

As explained at the beginning of this section (see also *appendix D*), at the level of $J = 2$, we have four states which couple to the Wilson loops. They correspond to the following scalar spherical harmonics on S^5

$$\begin{aligned} Y_{0,+2,0}^{2,2} &= \frac{1}{2} \cos^2 \vartheta \sin^2 \vartheta_2 e^{2i\varphi_2}, \quad Y_{0,-2,0}^{2,2} = \frac{1}{2} \cos^2 \vartheta \sin^2 \vartheta_2 e^{-2i\varphi_2}, \\ Y_{0,0,0}^{2,0} &= \frac{1}{2\sqrt{3}} (3 \sin^2 \vartheta - 1), \quad Y_{0,0,0}^{2,2} = -\frac{1}{2} \cos^2 \vartheta \cos 2\vartheta_2. \end{aligned} \quad (5.96)$$

On the string solution we have $\vartheta_2 = \pi/2, \varphi_2 = 0$, and so these harmonics reduce to

$$Y_{0,+2,0}^{2,2} = Y_{0,-2,0}^{2,2} = Y_{0,0,0}^{2,2} = \frac{1}{2} \cos^2 \vartheta, \quad Y_{0,0,0}^{2,0} \text{ unchanged.} \quad (5.97)$$

We find the following results (higher order results for $Y_{0,\pm J,0}^J$ for $J = 2, 3, 4$ have been reported in *appendix F*; here we are interested in the leading order in θ_0, θ_1 which is given by $J = 2$)

$$\begin{aligned} \left. \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \right|_{\frac{1}{2} \cos^2 \vartheta} &= \frac{\lambda}{8N^2} \left[\frac{\theta_0^3 \theta_1^3}{2^2} + \mathcal{O}(\theta^{10}) \right], \\ \left. \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \right|_{Y_{0,0,0}^{2,0}} &= \frac{\lambda}{10N^2} \left[\frac{13 \theta_0^4 \theta_1^4}{5 \cdot 3^2} + \frac{3(\theta_0^4 \theta_1^5 + \theta_0^5 \theta_1^4)}{2^4} + \mathcal{O}(\theta^{10}) \right], \end{aligned} \quad (5.98)$$

where $\mathcal{O}(\theta^n)$ is shorthand for terms of the form $\theta_0^p \theta_1^q$ where $p + q \geq 10$. The result coming from large- N YM_2 in the large λ, N limit calculated in section (3.17) is given by

$$\left. \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \right|_{\text{YM}_2} = \frac{\lambda}{8N^2} \left[\frac{\theta_0^3 \theta_1^3}{2^2} + \mathcal{O}(\theta^{10}) \right], \quad (5.99)$$

and so matches the contribution of **one** of the three modes $Y_{0,+2,0}^2, Y_{0,-2,0}^2, Y_{0,0,0}^{2'}$. The other two modes give contributions which left uncanceled spoil the agreement with YM_2 . The $Y_{0,0,0}^2$ mode contributes at subleading order, i.e. $\theta_0^4 \theta_1^4$, and so doesn't concern us here. In the next sections we will consider the fluctuations of the B-field which we will find also lead as $\theta_0^3 \theta_1^3$. However we will find that they do not remove the extra two contributions of the first line in (5.98).

NS-NS B-field on S^5

Continuing up the spectrum, the next lightest modes (outside of the s^J) stem from the fluctuation of the NS-NS B-field which can have both legs in either the S^5 , or the AdS_5 directions, see eq. (2.48) and what follows it in [100]. Here we treat the S^5 directions, whose fluctuations correspond to an AdS_5 scalar field

$$\delta B_{\alpha\beta} = a_-^k(x) Y_{[\alpha\beta]}^{k,-}(\Omega), \quad m_{a_-^k}^2 = k^2 - 4. \quad (5.100)$$

The conformal dimension Δ of an operator related to a scalar field on AdS_{d+1} with mass m is given by

$$\Delta = \frac{d}{2} + \sqrt{m^2 + \frac{d^2}{4}}. \quad (5.101)$$

Thus here we have

$$\Delta = k + 2. \quad (5.102)$$

The $k = 1$ mode thus corresponds to a gauge theory operator of dimension 3, in the **10** of $\text{SU}(4)$. Consulting appendix A of [103], we find that the operator is $\mathcal{E}_{AB} = \psi_A \psi_B$.

The antisymmetric tensor spherical harmonics $Y_{[\alpha\beta]}^{k,\pm}(\Omega)$ obey the following equations

$$\epsilon_{\alpha\beta}^{\gamma\delta\lambda} \Phi_\gamma Y_{[\delta\lambda]}^{k,\pm} = \pm 2i(k+2) Y_{[\alpha\beta]}^{k,\pm}, \quad (-\nabla_{S^5}^2 + 6) Y_{[\alpha\beta]}^{k,\pm} = (k+2)^2 Y_{[\alpha\beta]}^{k,\pm}, \quad (5.103)$$

and may be constructed using the (regular) tensor spherical harmonics given by

$$Y_{[\alpha\beta]}^k = \Phi_\alpha x^i \Phi_\beta x^j C_{[ijl_1](l_2 \dots l_k)} x^{l_1} \dots x^{l_k}, \quad (5.104)$$

where C is antisymmetric in i, j, l_1 , symmetric in l_2, \dots, l_k , and traceless on any pair of indices. Using the complex basis (D.49), the $Y_{[\alpha\beta]}^{k,\pm}$ amount to a choice of sign for the charges associated with the angles $\varphi, \varphi_2, \varphi_3$. As it turns out, our Wilson

loop couples only to $\delta B_{\vartheta\varphi}$ and so we require only $Y_{[\vartheta,\varphi]}^{1,-}$. There are only two modes, given by

$$Y_{[\vartheta,\varphi]}^{1,-} = \sin \vartheta \sin \vartheta_2 e^{-i\varphi_2}, \quad Y_{[\vartheta,\varphi]}^{1',-} = \sin \vartheta \cos \vartheta_2 e^{-i\varphi_3}, \quad (5.105)$$

where we have not yet normalized the spherical harmonics. Only the first will be non-zero on the string worldsheet.

The quadratic action for these fluctuations has been given in [110], see eq. (4.3) therein. One has

$$S = \frac{2}{2\kappa^2} \int d^{10}x \sqrt{g} \left(\frac{1}{2} \left[\Phi_\mu B_{\alpha\beta}^* \Phi^\mu B^{\alpha\beta} + \nabla_\gamma B_{\alpha\beta}^* \nabla^\gamma B^{\alpha\beta} + 6 B_{\alpha\beta}^* B^{\alpha\beta} \right] - i \epsilon^{\alpha\beta\gamma\delta\epsilon} B_{\alpha\beta}^* \Phi_\gamma B_{\delta\epsilon} \right), \quad (5.106)$$

where, in units where the radius of AdS_5 is unity, $1/(2\kappa^2) = 4N^2/(2\pi)^5$. Subbing-in (5.100), we find

$$S = 2C_k \frac{4N^2}{(2\pi)^5} \int_{AdS_5} d^5x \sqrt{g} \left(\frac{1}{2} \left[\Phi_\mu a^{k,-} \Phi^\mu a^{k,-} + m_{a^{k,-}}^2 \left(a^{k,-} \right)^2 \right] \right), \quad (5.107)$$

where the constant C_k encodes the normalization of the spherical harmonics. Specifically one has

$$C_1 = \int d\Omega_5 g^{\vartheta\vartheta} g^{\varphi\varphi} \left| Y_{[\vartheta,\varphi]}^{1,-} \right|^2 = \frac{\pi^3}{2}. \quad (5.108)$$

Thus the propagator is given by

$$P = \frac{\tilde{\alpha}_0}{\tilde{B}_k} W^\Delta {}_2F_1(\Delta, \Delta - 3/2, 2\Delta - 3; -4W) \quad (5.109)$$

where $\tilde{\alpha}_0 = (\Delta - 1)/(2\pi^2)$, and $\tilde{B}_k = 8N^2 C_k/(2\pi)^5$. See section 5.6 for the definition of W .

Coupling to the string worldsheet, we have

$$\begin{aligned} \delta S &= i \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \epsilon^{ab} \Phi_a X^M \Phi_b X^N \delta B_{MN} = -i \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \vartheta' \delta B_{\vartheta\varphi} \\ &= i \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \sin \vartheta \delta B_{\vartheta\varphi}, \end{aligned} \quad (5.110)$$

where a factor of i has been included due to the Euclidean signature of the worldsheet. Evaluating the contribution of the $k = 1$ mode to the correlator we find

$$\left. \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \right|_{\delta B_{\alpha\beta}} = -\frac{\lambda}{N^2} \frac{1}{2^4} \left(\frac{\theta_0^3 \theta_1^3}{8} - \frac{(\theta_0^3 \theta_1^4 + \theta_0^4 \theta_1^3)}{5} \right) + \mathcal{O}(\theta^8). \quad (5.111)$$

It is straightforward to further evaluate the $k = 2$ contributions. They lead as $\theta_0^4 \theta_1^4$ and so don't concern us here.

NS-NS B-field on AdS_5

The supergravity action for fluctuations of the NS-NS B field with both legs in the AdS_5 directions has been worked out in [110], while the dual gauge theory operator (for the lightest mode) has been discussed in [102]. The AdS/CFT correspondence relates linear combinations of the Ramond-Ramond 2-form potential $C_{\mu\nu}$ and the NS-NS B field $B_{\mu\nu}$ to dual operators in the gauge theory [110]

$$A = \sqrt{2}(B + iC), \quad \bar{A} = \sqrt{2}(B - iC), \quad B = \frac{1}{2\sqrt{2}}(A + \bar{A}), \quad C = \frac{1}{2\sqrt{2}i}(A - \bar{A}), \quad (5.112)$$

for which the action of the modes with both legs in the AdS_5 directions is given by

$$S = \int d^{10}x \sqrt{-g} \left(-\frac{1}{2} (\nabla_\mu \bar{A}_{\nu\rho} (\nabla^\mu A^{\nu\rho} - \nabla^\nu A^{\mu\rho} - \nabla^\rho A^{\nu\mu}) + \nabla_\alpha \bar{A}_{\mu\nu} \nabla^\alpha A^{\mu\nu}) \right. \\ \left. + i\epsilon^{\mu\nu\rho\tau\sigma} \bar{A}_{\mu\nu} \Phi_\rho A_{\tau\sigma} \right). \quad (5.113)$$

The equation of motion for $A_{\mu\nu}$ factorizes into two first order differential equations (c.f. eq. (2.61) in [100]),

$$\left[2k + i^*D \right] \left[2(k+4) - i^*D \right] A_{\mu\nu} = 0, \quad (5.114)$$

where *D is the operator $^*DA_{\mu\nu} = \epsilon_{\mu\nu}^{\rho\sigma\tau} \Phi_\rho A_{\sigma\tau}$. Thus $A_{\mu\nu}$ decomposes into two modes A_1 and A_2 which obey the two first order equations respectively. In order to realize this at the level of the action one must introduce auxilliary fields $P_{\mu\nu}$ and $\bar{P}_{\mu\nu}$ and write the action as [110]

$$S = \int d^{10}x \sqrt{-g} \left(-\frac{i}{2} \epsilon^{\mu\nu\rho\sigma\tau} \bar{P}_{\mu\nu} \Phi_\rho A_{\sigma\tau} + \frac{i}{2} \epsilon^{\mu\nu\rho\sigma\tau} P_{\mu\nu} \Phi_\rho \bar{A}_{\sigma\tau} \right. \\ \left. - 2\bar{P}_{\mu\nu} P^{\mu\nu} - \frac{1}{2} \nabla_\alpha \bar{A}_{\mu\nu} \nabla^\alpha A^{\mu\nu} + i\epsilon^{\mu\nu\rho\sigma\tau} \bar{A}_{\mu\nu} \Phi_\rho A_{\sigma\tau} \right), \quad (5.115)$$

and following another linear shift

$$A_1 = \frac{1}{2} (-\nabla_\alpha \nabla^\alpha + 4)^{\frac{1}{4}} A + (-\nabla_\alpha \nabla^\alpha + 4)^{-\frac{1}{4}} (P - A), \quad (5.116) \\ A_2 = \frac{1}{2} (-\nabla_\alpha \nabla^\alpha + 4)^{\frac{1}{4}} \bar{A} - (-\nabla_\alpha \nabla^\alpha + 4)^{-\frac{1}{4}} (\bar{P} - \bar{A}),$$

one gets the following action

$$S = - \int d^{10}x \sqrt{-g} \left(\frac{i}{2} \epsilon^{abcde} (\bar{A}_{1ab} \Phi_c A_{1de} + \bar{A}_{2ab} \Phi_c A_{2de}) \right. \\ \left. + (\sqrt{(-\nabla_\alpha \nabla^\alpha + 4)} + 2) \bar{A}_{1ab} A_1^{ab} + (\sqrt{(-\nabla_\alpha \nabla^\alpha + 4)} - 2) \bar{A}_{2ab} A_2^{ab} \right) \quad (5.117)$$

Expanding the fields in scalar spherical harmonics Y^k , one may replace the Laplacian on S^5 with $-k(k+4)$ yielding

$$\sqrt{(-\nabla_\alpha \nabla^\alpha + 4)} = k + 2, \quad k \geq 0, \quad (5.118)$$

and so A_2 is the lighter field. In fact the $k = 0$ mode is not physical and can be gauged away (see the text underneath eq. (2.63) in [100]). This leaves us with $k = 1$. This mode has been discussed in detail in the paper [102]. There it is argued that the dual CFT operator is

$$2i \Phi^{AB} F_{\mu\nu}^+ + \bar{\psi}^A \sigma_{\mu\nu} \bar{\psi}^B. \quad (5.119)$$

Bulk-to-bulk propagator

The bulk-to-bulk propagator for the field A_2 was given in [111]. The propagator is expressed as

$$\mathcal{P}_{\mu\nu;\bar{\mu}\bar{\nu}} = (G + 2H) T_{\mu\nu;\bar{\mu}\bar{\nu}}^1 + H' T_{\mu\nu;\bar{\mu}\bar{\nu}}^2 + K T_{\mu\nu;\bar{\mu}\bar{\nu}}^3, \quad (5.120)$$

where

$$G(u) = \frac{2^{3/2}}{8\pi^2} \frac{1}{[u(u+2)]^{3/2}}, \quad (5.121)$$

and where $u = 1/(2W)$ (W being given (5.92) of this document). Further, we have

$$K = G', \quad H = -(1+u)G' - 2G, \quad (5.122)$$

prime denoting differentiation by u . The tensors $T_{\mu\nu;\bar{\mu}\bar{\nu}}^i$ are given by

$$\begin{aligned} T_{\mu\nu;\bar{\mu}\bar{\nu}}^1 &= (\Phi_\mu \Phi_{\bar{\mu}} u) (\Phi_\nu \Phi_{\bar{\nu}} u) - (\Phi_\mu \Phi_{\bar{\nu}} u) (\Phi_\nu \Phi_{\bar{\mu}} u), \\ T_{\mu\nu;\bar{\mu}\bar{\nu}}^2 &= (\Phi_\mu u) (\Phi_{\bar{\mu}} u) (\Phi_\nu \Phi_{\bar{\nu}} u) - (\Phi_\nu u) (\Phi_{\bar{\mu}} u) (\Phi_\mu \Phi_{\bar{\nu}} u) \\ &\quad - (\Phi_\mu u) (\Phi_{\bar{\nu}} u) (\Phi_\nu \Phi_{\bar{\mu}} u) + (\Phi_\nu u) (\Phi_{\bar{\nu}} u) (\Phi_\mu \Phi_{\bar{\mu}} u), \\ T_{\mu\nu;\bar{\mu}\bar{\nu}}^3 &= \epsilon_{\mu\nu}{}^{\rho\lambda\sigma} (\Phi_\rho \Phi_{\bar{\mu}} u) (\Phi_\lambda \Phi_{\bar{\nu}} u) (\Phi_\sigma u). \end{aligned} \quad (5.123)$$

Coupling to string worldsheet

The string worldsheet couples to the B-field as per (5.110). Since our string solution in the AdS_5 directions has only the variable ϕ which depends on worldsheet- τ , and only y and r which depend on worldsheet- σ , we find

$$S = i \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \left(y' B_{\phi y} + r' B_{\phi r} \right), \quad (5.124)$$

where prime denotes differentiation by σ .

We are now faced with the task of relating the fluctuations of the B-field to the fluctuations of the physical propagating mode A_2 . We begin by considering the field redefinition (5.116). The auxiliary field \bar{P}_{ab} is defined by its equation of motion stemming from (5.115)

$$\bar{P}_{\mu\nu} = \frac{i}{4} \epsilon_{\mu\nu}{}^{\rho\lambda\sigma} \Phi_\rho \bar{A}_{\lambda\sigma}. \quad (5.125)$$

But, since we are interested only in the propagation of A_2 , the A field must also obey the first order equation of motion stemming from the first factor in (5.114), therefore

$$\frac{i}{4} \epsilon_{\mu\nu}{}^{\rho\lambda\sigma} \Phi_\rho \bar{A}_{\lambda\sigma} = -\frac{1}{2k} \bar{A}_{\mu\nu}. \quad (5.126)$$

By (5.116) we therefore have for the $k = 1$ mode

$$A_{2\mu\nu} = \frac{\sqrt{3}}{2} \bar{A}_{\mu\nu} - \frac{1}{\sqrt{3}} \left(\frac{i}{4} \epsilon_{\mu\nu}{}^{\rho\lambda\sigma} \Phi_\rho \bar{A}_{\lambda\sigma} - \bar{A}_{\mu\nu} \right) = \sqrt{3} \bar{A}_{\mu\nu} = \sqrt{3} \sqrt{2} B_{\mu\nu}. \quad (5.127)$$

The contributing $k = 1$ spherical harmonics are two,

$$Y_{0,1,0}^1 = \cos \vartheta_1 \sin \vartheta_2 e^{i\varphi_2}, \quad Y_{0,-1,0}^1 = \cos \vartheta_1 \sin \vartheta_2 e^{-i\varphi_2}, \quad (5.128)$$

and each give the same contribution to the correlator

$$\begin{aligned} \left. \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \right|_{\delta B_{\mu\nu}} &= -\frac{\lambda}{4\pi^2} \left(\frac{1}{\sqrt{2}\sqrt{3}} \right)^2 \frac{(2\pi)^5}{4N^2} \frac{3}{\pi^3} \int d\tau d\sigma \int d\bar{\tau} d\bar{\sigma} \cos \vartheta \cos \bar{\vartheta} \\ &\quad \times \left[y' \bar{y}' \mathcal{P}_{\phi y; \bar{\phi} \bar{y}} + r' \bar{r}' \mathcal{P}_{\phi r; \bar{\phi} \bar{r}} + y' \bar{r}' \mathcal{P}_{\phi y; \bar{\phi} \bar{r}} + r' \bar{y}' \mathcal{P}_{\phi r; \bar{\phi} \bar{y}} \right], \end{aligned} \quad (5.129)$$

where we have included the factor $1/(2\kappa^2)$ from outside the supergravity action giving $(2\pi)^5/(4N^2)$ and the normalization of the $k = 1$ spherical harmonic which is $\pi^3/3$. The result evaluates to (adding a factor of two to account for the two modes in (5.128))

$$\left. \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \right|_{\delta B_{\mu\nu}} = -\sqrt{2} \frac{\lambda}{N^2} \frac{1}{2^3} \left(\frac{3\theta_0^3 \theta_1^3}{8} + \frac{(\theta_0^3 \theta_1^4 + \theta_0^4 \theta_1^3)}{5} \right) + \mathcal{O}(\theta^8). \quad (5.130)$$

This result, in combination with (5.111), does not cancel the extra two contributions of the first line in (5.98) which spoil the agreement with YM_2 at the leading order.

Boundary terms

In the usual way of comparing two-point functions between supergravity and the CFT, the on-shell supergravity action is evaluated. However, for fields with single-derivative kinetic terms, like here, and also for fermions, the on-shell action vanishes identically. The solution has been to add boundary terms to the action. In this case the boundary term is [102][112]

$$S = \int d^9x \frac{1}{2} A_{ij} A^{ij}, \quad (5.131)$$

where i, j are indices on the boundary of AdS_5 . The natural question arises as to whether the presence of such a term could affect the bulk-to-bulk correlator computation done here. We believe it does not for the following reason. In our case the coupling to the boundary term is $r' B_{\phi r}$, but r' is zero at the boundary. Thus our Wilson loop has zero coupling to the boundary term.

Heavier modes

The modes we have considered correspond to gauge theory operators of dimension 2 and 3. Going one step higher in dimension, we have the dimension-3 chiral primaries, and at dimension-4 there are supergravity fluctuations of the dilaton field, massless symmetric-traceless tensor in AdS_5 (i.e. graviton), massless AdS_5 vector fluctuations (stemming from fluctuations of the $g_{\mu\alpha}$ metric components), and of course the higher KK-modes of the fluctuations computed here, i.e. the $k = 2$ modes of the B-field on S^5 and AdS_5 . With the exception of the $k = 2$ mode of the AdS_5 B-field, where the literature provides no bulk-to-bulk propagator, we have verified that all of these modes give contributions to the correlator which lead as $\theta_0^4 \theta_1^4$.

Chapter 6

Summary and Outlook

In this thesis some recent developments in $\mathcal{N}=4$ Supersymmetric Yang-Mills theory have been analyzed. In particular we have studied in detail a novel class of Wilson loops that lies on a three dimensional sphere. These operators are very nice observables since for particular configurations they are exactly calculable. Indeed when the contour lies on a two dimensional sphere there are strong evidence that their expectation value is captured by the analogous calculation in the zero instanton sector of the ordinary bosonic two dimensional Yang Mills theory.

The first original contribution of this work (sec. 3, [75]) has been a perturbative two-loop check of the conjecture. Interestingly we have found that, differently from the 1/2 BPS circle case, the interaction diagrams don't cancel but have an intriguing interplay with the ladder diagrams to reproduce exactly the matrix model governing YM_2 on S^2 . Since in literature other checks have been performed, for completeness we have reported them at the end of the same chapter.

The second original contribution of this thesis (sec. 4.1, [113]) has been a two-loop evaluation of another class of Wilson loops defined on a two dimensional Hyperbolic space. Similarly to the previous case, an exact agreement between the $\mathcal{N}=4$ SYM calculation and the matrix model that capture the YM_2 on \mathbb{H}_2 has been found.

The last original contribution (sec. 5, [78, 79]) has been the extension of the equivalence to the case of the connected correlators of Wilson loops. After the computation of the two matrix model governing the correlator in YM_2 , we have argued its equivalence with the DGRT Wilson loops Correlator in $\mathcal{N}=4$ SYM. In order to verify our proposal we have performed perturbative checks up to g^6 order and non-perturbative tests using the dual supergravity pictures.

There are several hints that deserve a future investigation. For example in our analysis only loops that lie on S^2 have been considered. However in the origi-

nal paper [60] more general operators on S^3 has been presented and thus it will be interesting to understand if and how they localize on some finite dimensional configurations.

Another aspect to be investigated is the generalized conjecture made by Pestun and Giombi [26] suggesting that the insertion of 1/2 BPS 't Hooft loops in $\mathcal{N} = 4$ SYM and their correlators with Wilson loops on S^2 are captured by the non-zero(unstable) instanton contributions to the partition function of the 2d Yang-Mills theory. It will be intriguing to check the proposal both at perturbative and non-perturbative level.

Furthermore since the S-duality predicts some magnetic objects preserving less supersymmetries than the 't Hooft circle loop (they should be dual to DGRT Wilson loops), it would be nice to define these operators and to study their properties. All these directions are currently under investigation.

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Appendix

A The Superconformal algebra $PSU(2, 2|4)$

In this section we write explicitly the $PSU(2, 2|4)$ algebra following the convention of [86]. In the theory there are eight Poincaré supercharges $Q_{\alpha\dot{a}}^a$, eight superconformal charges $S_{\alpha\dot{a}}^a$ and their conjugates $\bar{Q}_{\dot{a}}^{\dot{a}a}$ and $\bar{S}_{\dot{a}}^{\dot{a}a}$ ($\alpha, \dot{\alpha}, a, \dot{a} = 1, 2$). The original $SU(4)$ index I (Q^I) has been splitted into two $SU(2)$ spinor indices (a, \dot{a}). The dotted index \dot{a} belongs to the fundamental of $SU_a(2)$, while the second one a lives in the fundamental of $SU_b(2)$. The commutation relations are given by

$$\begin{aligned}
[K_{\alpha\dot{\alpha}}, Q_{\beta}^{\dot{a}a}] &= \epsilon_{\alpha\beta} \bar{S}_{\dot{\alpha}}^{\dot{a}a} & [K_{\alpha\dot{\alpha}}, \bar{Q}_{\beta\dot{a}}^a] &= \epsilon_{\dot{\alpha}\dot{\beta}} S_{\alpha}^a \\
[P_{\alpha\dot{\alpha}}, Q_{\beta}^{\dot{a}a}] &= \epsilon_{\alpha\beta} \bar{Q}_{\dot{\alpha}}^{\dot{a}a} & [P_{\alpha\dot{\alpha}}, \bar{S}_{\beta\dot{a}}^a] &= \epsilon_{\dot{\alpha}\dot{\beta}} Q_{\alpha}^a \\
[Q_{\alpha}^{\dot{a}a}, \bar{Q}_{\dot{\alpha}\dot{b}}^b]_+ &= -\epsilon^{\alpha\beta} \delta_{\dot{\beta}}^{\dot{a}} P_{\alpha\dot{\alpha}} & [S_{\alpha}^{\dot{a}a}, \bar{S}_{\dot{\alpha}\dot{b}}^b]_+ &= -\epsilon^{\alpha\beta} \delta_{\dot{\beta}}^{\dot{a}} K_{\alpha\dot{\alpha}} \\
[Q_{\alpha}^{\dot{a}a}, S_{\beta}^{\dot{a}a}]_+ &= \epsilon^{\dot{a}\dot{b}} \epsilon^{ab} J_{\alpha\beta} + \frac{1}{2} \epsilon_{\alpha\beta} \left(\epsilon^{ab} T^{\dot{a}\dot{b}} + \epsilon^{\dot{a}\dot{b}} T^{ab} - M^{\dot{a}\dot{b}ab} - \epsilon^{\dot{a}\dot{b}} \epsilon^{ab} \mathcal{D} \right) \\
[\bar{Q}_{\dot{\alpha}}^{\dot{a}a}, \bar{S}_{\beta\dot{b}}^b]_+ &= -\epsilon_{\dot{a}\dot{b}} \epsilon^{ab} \bar{J}_{\dot{\alpha}\dot{\beta}} + \frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} \left(\epsilon^{ab} \bar{T}_{\dot{a}\dot{b}} - \epsilon_{\dot{a}\dot{b}} T^{ab} - M_{\dot{a}\dot{b}}^{ab} + \epsilon_{\dot{a}\dot{b}} \epsilon^{ab} \mathcal{D} \right) \\
[T_b^a, Q_{\alpha}^{\dot{c}c}] &= -\delta_b^c Q_{\alpha}^{\dot{c}a} + \frac{1}{2} \delta_b^a Q_{\alpha}^{\dot{c}c} & [\bar{T}_{\dot{b}}^{\dot{a}}, \bar{Q}_{\alpha}^{\dot{c}c}] &= -\delta_{\dot{b}}^{\dot{c}} Q_{\alpha}^{\dot{a}c} + \frac{1}{2} \delta_{\dot{b}}^{\dot{a}} Q_{\alpha}^{\dot{c}c} \\
[M_{\dot{m}m}, Q_{\alpha}^{\dot{a}a}] &= -\frac{1}{2} (\tau_{\dot{m}})_{\dot{b}}^{\dot{a}} (\tau_m)_b^a Q_{\alpha}^{\dot{b}b} & [M_{\dot{m}m}, \bar{Q}_{\alpha\dot{a}}^a] &= -\frac{1}{2} (\tau_{\dot{m}})_{\dot{a}}^{\dot{b}} (\tau_m)_b^a \bar{Q}_{\alpha}^{\dot{b}b} \\
[M_{\dot{m}m}, S_{\alpha}^{\dot{a}a}] &= -\frac{1}{2} (\tau_{\dot{m}})_{\dot{b}}^{\dot{a}} (\tau_m)_b^a S_{\alpha}^{\dot{b}b} & [M_{\dot{m}m}, \bar{S}_{\alpha\dot{a}}^a] &= -\frac{1}{2} (\tau_{\dot{m}})_{\dot{a}}^{\dot{b}} (\tau_m)_b^a \bar{S}_{\alpha}^{\dot{b}b} \\
[\mathcal{D}, Q_{\alpha}^{\dot{a}a}] &= -Q_{\alpha}^{\dot{a}a} & [\mathcal{D}, \bar{Q}_{\alpha\dot{a}}^a] &= -\bar{Q}_{\alpha\dot{a}}^a \\
[\mathcal{D}, S_{\alpha}^{\dot{a}a}] &= +Q_{\alpha}^{\dot{a}a} & [\mathcal{D}, \bar{S}_{\alpha\dot{a}}^a] &= +\bar{S}_{\alpha\dot{a}}^a
\end{aligned}$$

where $\dot{T}_{\dot{b}}^{\dot{a}}$ (**1,3**), T_b^a (**3,1**) and $M_{\dot{b}b}^{\dot{a}a} = (\tau_{\dot{m}})_{\dot{b}}^{\dot{a}} (\tau_m)_b^a M_{\dot{m}m}$ (**3,3**) are the decomposition of the $SU(4)$ R -symmetry generators (**15**) under $SU(2)_a \times SU(2)_b$.

Here J_{β}^{α} and $\bar{J}_{\beta}^{\dot{\alpha}}$ are the generators of $SU(2)_L \times SU(2)_R$ Lorentz group, $P_{\alpha\dot{\alpha}}$ of the translations, $K^{\alpha\dot{\alpha}}$ of the conformal transformations and D of the dilatations. The other commutators are the canonical ones ($[D, \mathcal{O}] = \dim(\mathcal{O}) \mathcal{O}$ with $\dim(\mathcal{O})$ is the dimension of operator \mathcal{O}).

This $PSU(2, 2|4)$ algebra is precisely the algebra of the $(4|4) \times (4|4)$ complex supermatrix (a Z_2 -graded analog of the ordinary matrix).

B Yang-Mills theories in two dimensions

Two dimensional quantum chromodynamics is a good laboratory to obtain non-trivial information about quantum gauge theories. Several phenomena can be well understood in QCD_2 since the theory is exactly solvable. The basic reason is that gluons are not dynamical in two dimensions and as consequence the correlations

function depend only on the topology and the area of the manifold M on which the theory is defined. Furthermore in the large N -limit a string picture was derived [9] and the partition function [72, 74], Wilson loops [83] and field strength correlators [114] were computed exactly on a manifold with arbitrary genus.

Let's briefly review some general properties of this theory. We consider the partition function for a $SU(N)$ gauge theory on a manifold M

$$\mathcal{Z}_M = \int \mathcal{D}A_\mu \exp \left(-\frac{1}{4g^2} \int_M d^2x \sqrt{g} \text{Tr} F_{\mu\nu} F^{\mu\nu} \right) \quad (\text{B.1})$$

The theory is invariant under all diffeomorphisms that preserving the area. Indeed since the two dimensional field $F_{\mu\nu}$ can be uniquely written as

$$F_{\mu\nu} = \sqrt{g} \epsilon_{\mu\nu} f \quad (\text{B.2})$$

where $\epsilon_{\mu\nu}$ is the anty-simmetric tensor and f is a scalar field, the action (B.1) reads

$$\mathcal{Z}_M = \int \mathcal{D}A_\mu \exp \left(-\frac{1}{4g^2} \int_M d^2x \sqrt{g} \text{Tr} f^2 \right) \quad (\text{B.3})$$

and so the metric appears only trough the volume $d^2x \sqrt{g}$. As consequence a diffeomorphism that leaves invariant the area, doesn't change (B.3).

The next step is to evaluate this partition function. To do this we follow the idea of Migdal [72] to use a lattice regularization of the theory. So (B.1) in this regularization can be written as

$$\mathcal{Z}_M = \int \prod_L dU_L \prod_{\text{plaquettes}} \mathcal{Z}_P[U_P] \quad (\text{B.4})$$

where \mathcal{Z}_P is the heat kernel action of the plaquette defined as

$$\mathcal{Z}_P = \sum_R d_R \chi_R(U_P) e^{-g^2 C_2(R) A_p} \quad (\text{B.5})$$

with $\chi_R(U_P)$ the character of U_P in the representation R , A_p the area of the plaquette and $C_2(R)$ and d_R respectively the quadratic Casimir and the dimension of the representation R . Since the action \mathcal{Z}_P is additive the partition function will be independent of the triangulation. Using this fact one can rewrite the total action as

$$\mathcal{Z}_M = \sum_R d_R^{-2g} \exp \left(-\frac{\lambda A}{N} C_2(R) \right) \quad (\text{B.6})$$

that depends only on the area A and on the topology of the two dimensional surface (g is the genus of the manifold).

To calculate the expectation value of an observable it is useful to recall the definition the heat-kernel propagator

$$\mathcal{K}(A; U_1, U_2) = \langle U_2 | e^{-\frac{g^2 A \Delta}{2}} | U_1 \rangle = \sum_R \chi_R(U_1) \chi_R^\dagger(U_2) e^{-\frac{g^2 A}{2} C_2(R)}, \quad (\text{B.7})$$

where A is the area of the cylinder and the sum runs over all the representations R of $U(N)$. Using the disc amplitude, namely the cylinder (B.7) in which one of the two holonomies is trivial, it is straightforward to write down the VEV of a non-intersecting Wilson loop, for example, on a two sphere

$$\langle W \rangle_{A_1, A_2} = \frac{1}{\mathcal{Z}} \frac{1}{N} \int dU \text{Tr}[U] \mathcal{K}(A_1; I, U) \mathcal{K}(A_2; U, I). \quad (\text{B.8})$$

There is a dual representation of this quantity. In the ordinary YM_2 there are non trivial instanton solutions of the Yang-Mills equation labelled by a set of integer $\vec{n} = (n_1, \dots, n_N)$, $A_\mu = A_\mu^0 T_{\vec{n}}$ where A_μ^0 is the usual Dirac Potential. Eq.(B.8) can be interpreted as a sum over these solutions if we perform a Poisson resummation (see eq.7-9 of [71]). We are mainly interested to the restriction of this result to the zero-instanton sector, namely $n_i = 0$. Following [71] in this case the expectation value of the operator is given by

$$\begin{aligned} \langle W \rangle_{A_1, A_2} &= (2\pi)^{\frac{N}{2}} \prod_{n=0}^N \frac{1}{n!} \exp \left[-\frac{g^2 A_1 A_2}{2A} \right] \int_{-\infty}^{+\infty} dz_1 dz_2 \dots dz_N \exp \left[-\frac{1}{2} \sum_{i=1}^N z_i^2 \right] \times \\ &\times \prod_{j=2}^N \left[(z_1 - z_j)^2 + i\sqrt{g^2 A} (z_1 - z_j) - g^2 \frac{A_1 A_2}{A} \right] \Delta^2(z_2, \dots, z_N). \end{aligned} \quad (\text{B.9})$$

With a change of variables and if we express as usual the Van der Monde determinant in terms of the Hermite polynomials, the integral over the eigenvalues has been done and the result reads

$$\langle W \rangle_{A_1, A_2} = \frac{1}{N} \exp \left[-\frac{g^2 A_1 A_2}{2A} \right] L_{N-1}^1 \left(g^2 \frac{A_1 A_2}{A} \right) \quad (\text{B.10})$$

Another interesting features of the theory as anticipated in the introduction is its connection with the string theory. Following the original 't Hooft idea, Gross and Taylor try to formulate a string theory description of the QCD_2 [8, 9]. They didn't found a precise formulation of the dual string action but relate the expansion of the

QCD_2 free energy \mathcal{F} to a specific sum over maps. Indeed expanding \mathcal{F} in powers of $1/N$

$$\mathcal{F} = \sum_{g=0}^{\infty} N^{2-2g} f_g^G(\lambda A) \quad (\text{B.11})$$

one can interpret the coefficients $f_g^G(\lambda A)$ as the contribution of a map of a manifold with genus g manifold \mathcal{M}_g onto a manifold \mathcal{M}_G with genus G . More in detail f_g^G reads as

$$f_g^G(\lambda A) = \sum_n \sum_i \omega_{g,G}^{n,i} \exp \left[-\frac{n\lambda A}{2} \right] (\lambda A)^i \quad (\text{B.12})$$

where the sum over n is the sum over maps that wind n times the manifold \mathcal{M}_G and $\omega_{g,G}^{n,i}$ is expressed in terms of the number of topological continuous map, branch-points and collapsed handle.

C Useful Functions and Integrals

$$\boxed{\mathcal{I}_1(\mathbf{x}, \mathbf{y}) \text{ and } \mathcal{I}_2(\mathbf{x}, \mathbf{y})} \quad :$$

For the integral $\mathcal{I}_1(x, y)$ defined in section (3.3) one can easily perform the integration over the momenta. In order to integrate over p_1 , we first introduce the Feynman parametrization for the two denominators, which depends on p_1 . Then we perform the change of variable $p_1 \mapsto p_1 - \alpha p_2$. This yields

$$\begin{aligned} \mathcal{I}_1(x, y) &\equiv \int \frac{d^{2\omega} p_1 d^{2\omega} p_2}{(2\pi)^{4\omega}} \frac{e^{ip_1 x + ip_2 y}}{p_1^2 p_2^2 (p_1 + p_2)^2} = \\ &= \int_0^1 d\alpha \int \frac{d^{2\omega} p_2}{(2\pi)^{2\omega}} \frac{e^{ip_2(y-\alpha x)}}{p_2^2} \int \frac{d^{2\omega} p_1}{(2\pi)^{2\omega}} \frac{e^{ip_1 x}}{[p_1^2 + \alpha(1-\alpha)p_2^2]^2} \end{aligned} \quad (\text{C.13})$$

The integral over p_1 can be now evaluated by means of the Schwinger representation for the denominator in (C.13). We obtain

$$\begin{aligned} \mathcal{I}_1(x, y) &= \frac{1}{(4\pi)^\omega} \int_0^1 d\alpha \int_0^\infty d\beta \beta^{1-\omega} \int \frac{d^{2\omega} p_2}{(2\pi)^{2\omega}} \frac{e^{ip_2(y-\alpha x)}}{p_2^2} e^{-\frac{x^2}{4\beta} - \beta\alpha(1-\alpha)p_2^2} = \\ &= \frac{1}{(4\pi)^\omega} \int_0^1 d\alpha (\alpha(1-\alpha))^{\omega-2} \int_0^\infty d\beta \beta^{1-\omega} \int \frac{d^{2\omega} p_2}{(2\pi)^{2\omega}} \frac{e^{ip_2(y-\alpha x)}}{p_2^2} e^{-\frac{x^2}{4\beta} - \beta p_2^2}. \end{aligned} \quad (\text{C.14})$$

The integral over the second momentum can be now performed by introducing a second Schwinger parameter λ . We end up with the following parametric representation for $\mathcal{I}_1(x, y)$

$$\mathcal{I}_1(x, y) = \frac{1}{(4\pi)^{2\omega}} \int_0^1 d\alpha (\alpha(1-\alpha))^{\omega-2} \int_0^\infty d\beta \beta^{1-\omega} \int_0^\infty d\lambda (\lambda+\beta)^{-\omega} e^{-\frac{(y-\alpha x)^2}{4(\lambda+\beta)} - \frac{x^2 \alpha(1-\alpha)}{4\beta}}. \quad (\text{C.15})$$

By setting $\tau = \lambda + \beta$, we can first integrate over β and then over τ . In fact

$$\begin{aligned} \mathcal{I}_1(x, y) &= \frac{1}{(4\pi)^{2\omega}} \int_0^1 d\alpha (\alpha(1-\alpha))^{\omega-2} \int_0^\infty d\tau \tau^{-\omega} \int_0^\tau d\beta \beta^{1-\omega} e^{-\frac{(y-\alpha x)^2}{4\tau} - \frac{x^2 \alpha(1-\alpha)}{4\beta}} = \\ &= \frac{4^{\omega-2} (x^2)^{2-\omega}}{(4\pi)^{2\omega}} \int_0^1 d\alpha \int_0^\infty d\tau \tau^{\omega-2} e^{-\frac{(y-\alpha x)^2}{4}\tau} \Gamma\left(\omega-2, \frac{x^2(1-\alpha)\alpha\tau}{4}\right) = \\ &= \frac{\Gamma(2\omega-3)}{64\pi^{2\omega}(\omega-1)} \int_0^1 d\alpha \frac{[\alpha(1-\alpha)]^{\omega-2}}{[\alpha(x-y)^2 + (1-\alpha)y^2]^{2\omega-3}} \times \\ &\quad \times {}_2F_1\left(1, 2\omega-3, \omega, \frac{(y-\alpha x)^2}{\alpha(x-y)^2 + (1-\alpha)y^2}\right). \end{aligned} \quad (\text{C.16})$$

In the last equality, we have used the following integral given on the table

$$\int_0^\infty x^{\mu-1} e^{-\beta x} \Gamma(\nu, \alpha x) dx = \frac{\alpha^\nu \Gamma(\mu+\nu)}{\mu(\alpha+\beta)^{\mu+\nu}} {}_2F_1\left(1, \mu+\nu, \mu+1, \frac{\beta}{\beta+\alpha}\right). \quad (\text{C.17})$$

This representation is useful to study the behavior around $x=0$, $y=0$ and $y=x$. Since (C.13) is manifestly symmetric in the exchange $x \leftrightarrow y$ and $x \leftrightarrow y-x$, it is sufficient to study the behavior only around $x=0$. The other two cases will obviously display a similar behavior. At $x=0$ we find

$$\begin{aligned} \mathcal{I}_1(0, y) &= \frac{\Gamma(2\omega-3) {}_2F_1(1, 2\omega-3, \omega, 1)}{64\pi^{2\omega}(\omega-1) [y^2]^{2\omega-3}} \int_0^1 d\alpha [\alpha(1-\alpha)]^{\omega-2} = \\ &= \frac{\Gamma^2(\omega-1)}{64\pi^{2\omega}(2\omega-3)(2-\omega)} \frac{1}{[y^2]^{2\omega-3}} \end{aligned} \quad (\text{C.18})$$

The integral $\mathcal{I}_2(x, y)$ is defined as follows

$$\begin{aligned} \mathcal{I}_2(x, y) &= -\frac{\Gamma(2\omega-3)}{64\pi^{2\omega}(\omega-1)} \int_0^1 d\alpha \frac{\alpha^{\omega-1} (1-\alpha)^{\omega-2}}{[\alpha(1-\alpha)x^2 + (y-\alpha x)^2]^{2\omega-3}} \times \\ &\quad \times {}_2F_1\left(1, 2\omega-3, \omega, \frac{(y-\alpha x)^2}{(y-\alpha x)^2 + \alpha(1-\alpha)x^2}\right). \end{aligned} \quad (\text{C.19})$$

The origin of this object is explained in appendix A and it is related to light-cone gauge analysis of the Wilson-loop. There, its definition is given in momentum

space. The expression (C.19) is obtained performing the integration over the momenta along the same path followed for $\mathcal{I}_1(x, y)$.

In the following we shall compute its behavior at $x = 0$, $y = 0$ and $y = x$. At $x = 0$:

$$\begin{aligned}\mathcal{I}_2(x, y) &= -\frac{\Gamma(2\omega - 3)_2F_1(1, 2\omega - 3, \omega, 1)}{64\pi^{2\omega}[(\tilde{y})^2]^{2\omega-3}(\omega - 1)}\int_0^1 d\alpha \alpha^{\omega-1}(1 - \alpha)^{\omega-2} = \\ &= -\frac{\Gamma^2(\omega - 1)}{128\pi^{2\omega}(2 - \omega)(2\omega - 3)[(y)^2]^{2\omega-3}} = \\ &= \frac{1}{128\pi^4(\omega - 2)y^2} + O((\omega - 2)^0)\end{aligned}\tag{C.20}$$

At $y = 0$:

$$\begin{aligned}\mathcal{I}_2(x, y) &= -\frac{\Gamma(2\omega - 3)}{64\pi^{2\omega}(\omega - 1)[x]^{2\omega-3}}\int_0^1 d\alpha \alpha^{2-\omega}(1 - \alpha)^{\omega-2} {}_2F_1(1, 2\omega - 3, \omega, \alpha) = \\ &= -\frac{\Gamma(2\omega - 3)\Gamma(3 - \omega)\Gamma(\omega - 1)}{64\pi^{2\omega}(\omega - 1)[x]^{2\omega-3}} {}_3F_2(1, 2\omega - 3, 3 - \omega; \omega, 2|1) = \\ &= -\frac{\Gamma(2\omega - 3)\Gamma(3 - \omega)\Gamma(\omega - 1)}{64\pi^{2\omega}[x]^{2\omega-3}} \frac{(\Gamma(\omega - 2) - 2\Gamma(3 - \omega)\Gamma(2\omega - 4))}{4(\omega - 2)^3\Gamma(2 - \omega)\Gamma(2\omega - 4)} = \\ &= -\frac{1}{384\pi^2x^2} + O((\omega - 2)^1)\end{aligned}\tag{C.21}$$

At $y = x$:

$$\begin{aligned}\mathcal{I}_2 &= -\frac{\Gamma(2\omega - 3)}{64\pi^{2\omega}(\omega - 1)[x^2]^{2\omega-3}}\int_0^1 d\alpha \alpha^{\omega-1}(1 - \alpha)^{1-\omega} \times {}_2F_1(1, 2\omega - 3, \omega, 1 - \alpha) = \\ &= -\frac{\Gamma(2\omega - 3)\Gamma(2 - \omega)\Gamma(\omega)}{64\pi^{2\omega}(\omega - 1)[x^2]^{2\omega-3}} {}_3F_2(1, 2\omega - 3, 2 - \omega; \omega, 2|1) = \\ &= -\frac{\Gamma(2\omega - 3)\Gamma(2 - \omega)\Gamma(\omega)}{64\pi^{2\omega}(\omega - 1)[x^2]^{2\omega-3}} \frac{1 - \frac{\Gamma(\omega-1)}{\Gamma(3-\omega)\Gamma(2\omega-2)}}{2(\omega - 2)} = \\ &= \frac{1}{64\pi^4(\omega - 2)x^2} + O((\omega - 2)^0)\end{aligned}\tag{C.22}$$

A useful combination of \mathcal{I}_1 and \mathcal{I}_2

 :

In the following we shall show that the following combination of the derivatives of \mathcal{I}_1 and \mathcal{I}_2 ,

$$V_\mu = \frac{\partial \mathcal{I}_1(x, y)}{\partial x^\mu} - \frac{\partial \mathcal{I}_2(x, y)}{\partial y^\mu},\tag{C.23}$$

can be reduced to a very simple form. First, we shall take the derivative. We find

$$\begin{aligned} V_\mu &= \frac{\Gamma(2\omega-3)}{64\pi^{2\omega}(\omega-1)} \int_0^1 d\alpha \left(\frac{\partial}{\partial x^\mu} + \alpha \frac{\partial}{\partial y^\mu} \right) \left[\frac{[\alpha(1-\alpha)]^{\omega-2}}{[(\tilde{y}-\alpha\tilde{x})^2]^{2\omega-3}} G[\xi] \right] = \\ &= \frac{\Gamma(2\omega-3)}{64\pi^{2\omega}(\omega-1)} \int_0^1 d\alpha \frac{[\alpha(1-\alpha)]^{\omega-2}}{[(\tilde{y}-\alpha\tilde{x})^2]^{2\omega-3}} G'[\xi] \left(\frac{\partial \xi}{\partial x^\mu} + \alpha \frac{\partial \xi}{\partial y^\mu} \right), \end{aligned} \quad (\text{C.24})$$

where

$$\xi = \frac{(\tilde{y}-\alpha x)^2}{\alpha(1-\alpha)x^2 + (\tilde{y}-\alpha x)^2} \quad \text{and} \quad G[\xi] = \xi^{2\omega-3} {}_2F_1(1, 2\omega-3, \omega, \xi). \quad (\text{C.25})$$

Since

$$\left(\frac{\partial \xi}{\partial x^\mu} + \alpha \frac{\partial \xi}{\partial y^\mu} \right) = -2(1-\xi)\xi \frac{x^\mu}{x^2}, \quad (\text{C.26})$$

the expression for V_μ can be rewritten as follows

$$V_\mu = -\frac{2\Gamma(2\omega-3)\tilde{x}^\mu}{64\pi^{2\omega}(\omega-1)(\tilde{x}^2)^{2\omega-2}} \int_0^1 d\alpha [\alpha(1-\alpha)]^{1-\omega} G'[\xi] \xi^{4-2\omega} (1-\xi)^{2\omega-2}. \quad (\text{C.27})$$

The derivative of $G[\xi]$ can be now computed by using the well-known properties of the hypergeometric functions:

$$\begin{aligned} G'(\xi) &= \frac{2\omega-3}{\omega} \xi^{2\omega-4} (\omega {}_2F_1(1, 2\omega-3, \omega, \xi) + \xi {}_2F_1(2, 2\omega-2; \omega+1; \xi)) = \\ &= (2\omega-3) \xi^{2\omega-4} {}_2F_1(1, 2\omega-2; \omega; \xi) \end{aligned} \quad (\text{C.28})$$

where we have used the identity

$$\gamma_2 F_1(\alpha, \beta; \gamma; \xi) - \gamma_2 F_1(\alpha, \beta+1; \gamma; \xi) + \alpha \xi {}_2F_1(\alpha+1, \beta+1; \gamma+1; \xi) = 0. \quad (\text{C.29})$$

Thus

$$V_\mu = -\frac{\Gamma(2\omega-2)x^\mu}{32\pi^{2\omega}(\omega-1)(x^2)^{2\omega-2}} \int_0^1 d\alpha [\alpha(1-\alpha)]^{1-\omega} {}_2F_1(1, 2\omega-2; \omega; \xi) (1-\xi)^{2\omega-2}. \quad (\text{C.30})$$

$$\boxed{\mathcal{I}^{(4)}(x_1, x_2, x_3, x_4)} \quad :$$

We consider the integral

$$\mathcal{I}^{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{(4\pi^2)^4} \int \frac{d^4 z}{(x_1-z)^2 (x_2-z)^2 (y_1-z)^2 (y_2-z)^2}. \quad (\text{C.31})$$

It is well-known that this integral can be computed in terms of \mathcal{I}_1 [?]. In fact if we define

$$\bar{x}_1^\mu = \frac{(x_1-y_2)^\mu}{(x_1-y_2)^2}, \quad \bar{x}_2^\mu = \frac{(x_2-y_2)^\mu}{(x_2-y_2)^2}, \quad \bar{x}_3^\mu = \frac{(y_1-y_2)^\mu}{(y_1-y_2)^2}, \quad (\text{C.32})$$

we find

$$\mathcal{I}^{(4)}(x_1, x_2, x_3, x_4) = \frac{\bar{x}_1^2 \bar{x}_2^2 \bar{x}_3^2}{(4\pi^2)^4} \int \frac{d^4 z}{(\bar{x}_1 - z)^2 (\bar{x}_2 - z)^2 (\bar{x}_3 - z)^2} = \frac{\bar{x}_1^2 \bar{x}_2^2 \bar{x}_3^2}{4\pi^2} \mathcal{I}_1(\bar{x}_1 - \bar{x}_2, \bar{x}_3 - \bar{x}_2). \quad (\text{C.33})$$

Then

$$\begin{aligned} \mathcal{I}^{(4) \text{sing.}}(x_1, x_2, y_1, y_2) &= -\frac{\log(x_1 - x_2)^2}{256\pi^6} \times \\ &\times \int_0^1 \frac{d\alpha}{(1-\alpha)(y_1 - x_2)^2 (y_2 - x_1)^2 - \alpha(1-\alpha)(x_1 - x_2)^2 (y_1 - y_2)^2 + \alpha(y_1 - x_1)^2 (y_2 - x_2)^2}. \end{aligned} \quad (\text{C.34})$$

For our goals, the most convenient way to compute the integral \mathcal{S}^μ defined in (5.83) is to use the technique of [?], which allows us to reduce the tensor integrals to scalar integrals in higher space-time dimensions. We shall perform this reduction in 2ω dimensions and for arbitrary powers of the denominators. The final result is very nice and compact

$$\prod_{i=1}^4 \frac{\Gamma(a_i)}{4\pi^{a_i+1}} \int \frac{w^\mu d^{2\omega} w}{((x_1 - w)^2)^{a_1} ((x_2 - w)^2)^{a_2} ((x_3 - w)^2)^{a_3} ((x_4 - w)^2)^{a_4}} = \sum_{j=1}^4 x_j^\mu \mathfrak{S}(\omega + 1; a_i + \delta_{ij}) \quad (\text{C.35})$$

where

$$\mathcal{S}^{(2\omega)}(\omega; a_i) = \prod_{i=1}^4 \frac{\Gamma(a_i)}{4\pi^{a_i+1}} \int \frac{d^{2\omega} w}{((x_1 - w)^2)^{a_1} ((x_2 - w)^2)^{a_2} ((x_3 - w)^2)^{a_3} ((x_4 - w)^2)^{a_4}}. \quad (\text{C.36})$$

In computing \mathbf{A}_1 we also need the derivative with respect to x_2^ν of the above expression. After some manipulation this derivative can be arranged as follows

$$\begin{aligned} &\prod_{i=1}^4 \frac{\Gamma(a_i)}{4\pi^{a_i+1}} \frac{\partial}{\partial x_2^\nu} \int \frac{w^\mu d^{2\omega} w}{((x_1 - w)^2)^{a_1} ((x_2 - w)^2)^{a_2} ((x_3 - w)^2)^{a_3} ((x_4 - w)^2)^{a_4}} = \\ &= \delta^{\mu\nu} \mathfrak{S}(\omega + 1; a_i + \delta_{i2}) + 2\pi \sum_{k=1}^4 \sum_{j=1}^4 x_j^\mu (x_k - x_2)^\nu \mathfrak{S}(\omega + 2; a_i + \delta_{ij} + \delta_{i2} + \delta_{ki}). \end{aligned} \quad (\text{C.37})$$

Finally, the only other ingredient necessary for our calculation is the behavior of the integral $\mathcal{S}(2\omega; a_i)$ when x_1 and x_2 approach the same point x_0 .

$$\begin{aligned} \mathcal{S}(2\omega; a_i) = & \frac{\Gamma(\omega - a_1) \Gamma(\omega - a_2) \Gamma(a_3) \Gamma(a_4) \Gamma(a_1 + a_2 - \omega)}{256 \pi^{\sum_{i=1}^4 a_i + 4 - \omega}} \frac{((x_1 - x_2)^2)^{(\omega - a_1 - a_2)}}{((x_3)^2)^{a_3} ((x_4)^2)^{a_4}} \times \\ & \times \left[1 + 2 \left(a_3 \frac{x_3}{x_3^2} + a_4 \frac{x_4}{x_4^2} \right) \cdot \left((x_2 - x_0) + \frac{\omega - a_2}{2\omega - a_1 - a_2} (x_1 - x_2) \right) + O((x_1 - x_2)^2) \right]. \end{aligned} \quad (\text{C.38})$$

I functions

$$I(\delta) = \frac{2\pi}{\sqrt{(1 + |w|^2 - 2w_3 \cos \delta)^2 - 4(w_1^2 + w_2^2) \sin^2 \delta}} \quad (\text{C.39})$$

$$I_c(\delta) = \frac{2\pi w_1 \sin \delta}{4(w_1^2 + w_2^2)} \left(\frac{(1 + |w|^2 - 2w_3 \cos \delta)}{\sqrt{(1 + |w|^2 - 2w_3 \cos \delta)^2 - 4(w_1^2 + w_2^2) \sin^2 \delta}} - 1 \right) \quad (\text{C.40})$$

$$I_s(\delta) = \frac{2\pi w_2 \sin \delta}{4(w_1^2 + w_2^2)} \left(\frac{(1 + |w|^2 - 2w_3 \cos \delta)}{\sqrt{(1 + |w|^2 - 2w_3 \cos \delta)^2 - 4(w_1^2 + w_2^2) \sin^2 \delta}} - 1 \right) \quad (\text{C.41})$$

Here $|w|^2 = w_1^2 + w_2^2 + w_3^2 + w_4^2$.

D Spherical harmonics on S^5

We describe the metric of S^5 as follows

$$ds^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2 + \cos^2 \vartheta (d\vartheta_2^2 + \sin^2 \vartheta_2 d\varphi_2^2 + \cos^2 \vartheta_2 d\varphi_3^2), \quad (\text{D.42})$$

where $\vartheta, \vartheta_2 \in [0, \pi/2]$ and $\varphi, \varphi_2, \varphi_3 \in [0, 2\pi)$. The Laplacian is given by

$$\begin{aligned} \nabla^2 = & \Phi_\vartheta^2 - (3 \tan \vartheta - \cot \vartheta) \Phi_\vartheta + \csc^2 \vartheta \Phi_\varphi^2 \\ & + \sec^2 \vartheta (\Phi_{\vartheta_2}^2 + (\cot \vartheta_2 - \tan \vartheta_2) \Phi_{\vartheta_2} + \csc^2 \vartheta_2 \Phi_{\varphi_2}^2 + \sec^2 \vartheta_2 \Phi_{\varphi_3}^2). \end{aligned} \quad (\text{D.43})$$

The weight J scalar spherical harmonics obey $\nabla^2 Y^J = -J(J+4)Y^J$. This partial differential equation is separable and solvable. The orthogonal, but unnormalized solutions are given by

$$\begin{aligned} Y_{j_1, j_2, j_3}^{J, n} = & w^{|j_2|} (1 + w^2)^{1+n/2} z^{|j_1|} (1 + z^2)^{2+J/2} e^{i(j_1\varphi + j_2\varphi_2 + j_3\varphi_3)} \\ & {}_2F_1 \left(1 + \frac{1}{2}(J + |j_1| - n), 2 + \frac{1}{2}(J + |j_1| + n); 1 + |j_1|, -z^2 \right) \\ & {}_2F_1 \left(1 + \frac{1}{2}(|j_2| - |j_3| + n), 1 + \frac{1}{2}(|j_2| + |j_3| + n); 1 + |j_2|, -w^2 \right), \end{aligned} \quad (\text{D.44})$$

where $z = \tan \vartheta$ and $w = \tan \vartheta_2$, and

$$\begin{aligned} j_i &\in [-J, J], \quad J - \sum_i |j_i| = 0, 2, 4, \dots, J^{\text{even}}, \quad J^{\text{even}} = \begin{cases} J-1, & J \text{ odd} \\ J, & J \text{ even} \end{cases}, \\ n &= J - |j_1|, J - |j_1| - 2, \dots, |j_2| + |j_3|, \end{aligned} \quad (\text{D.45})$$

giving the requisite $(3+J)(2+J)^2(1+J)/12$ states, i.e. the number of components in a traceless symmetric rank- J tensor $C_{(l_1 \dots l_J)}$ in the embedding space R^6 , where the spherical harmonics may be expressed as

$$Y^J = C_{(l_1 \dots l_J)} x^{l_1} \dots x^{l_J}, \quad (\text{D.46})$$

where

$$\begin{aligned} x^1 &= \sin \vartheta \cos \varphi, \quad x^2 = \sin \vartheta \sin \varphi, \quad x^3 = \cos \vartheta \sin \vartheta_2 \cos \varphi_2, \\ x^4 &= \cos \vartheta \sin \vartheta_2 \sin \varphi_2, \quad x^5 = \cos \vartheta \cos \vartheta_2 \cos \varphi_3, \quad x^6 = \cos \vartheta \cos \vartheta_2 \sin \varphi_3. \end{aligned} \quad (\text{D.47})$$

The normalization of the $Y_{j_1, j_2, j_3}^{J, n}$ may be fixed using

$$\begin{aligned} \int_{S^5} \left| Y_{j_1, j_2, j_3}^{J, n} \right|^2 &= \\ &= 2\pi^3 \frac{(|j_1|!)^2 (|j_2|!)^2}{(J+2)(n+1)} \frac{\Gamma(1 + \frac{1}{2}(J - |j_1| - n))}{\Gamma(1 + \frac{1}{2}(J + |j_1| - n))} \frac{\Gamma(2 + \frac{1}{2}(J - |j_1| + n))}{\Gamma(2 + \frac{1}{2}(J + |j_1| + n))} \\ &\quad \times \frac{\Gamma(1 + \frac{1}{2}(-|j_2| - |j_3| + n))}{\Gamma(1 + \frac{1}{2}(|j_2| - |j_3| + n))} \frac{\Gamma(1 + \frac{1}{2}(-|j_2| + |j_3| + n))}{\Gamma(1 + \frac{1}{2}(|j_2| + |j_3| + n))}. \end{aligned} \quad (\text{D.48})$$

A more convenient basis for the presentation of the scalar spherical harmonics are the complex variables

$$z_1 = \sin \vartheta e^{i\varphi}, \quad z_2 = \cos \vartheta \sin \vartheta_2 e^{i\varphi_2}, \quad z_3 = \cos \vartheta \cos \vartheta_2 e^{i\varphi_3}. \quad (\text{D.49})$$

Using these the 6 Y^1 are given simply by $\{z_1, z_2, z_3, z_1^*, z_2^*, z_3^*\}$, while the 20 Y^2 may be summarized as

$$\begin{aligned} &\{z_1^2, z_2^2, z_3^2, z_1 z_2, z_1 z_3, z_2 z_3, z_1 z_2^*, z_1 z_3^*, z_2 z_3^*\} + \text{c.c.} \\ &\text{and } \{3|z_1|^2 - 1, |z_2|^2 - |z_3|^2\}. \end{aligned} \quad (\text{D.50})$$

On our string solution we have $\vartheta_2 = \pi/2$ and $\varphi_2 = \varphi_3 = 0$, which means $z_3 = 0$. However, there is a further simplification: the $U(1)$ symmetry of the string worldsheets parameterized by the angle φ implies that the contribution to the correlator is zero unless the Y^J are independent of φ . This issue has been discussed

in some detail in [107]. This leaves the following Y^2 harmonics (normalized in accordance with (5.93)¹)

$$\begin{aligned} Y_{0,+2,0}^{2,2} &= \frac{1}{2} \cos^2 \vartheta \sin^2 \vartheta_2 e^{2i\varphi_2}, & Y_{0,-2,0}^{2,2} &= \frac{1}{2} \cos^2 \vartheta \sin^2 \vartheta_2 e^{-2i\varphi_2}, \\ Y_{0,0,0}^{2,0} &= \frac{1}{2\sqrt{3}} (3 \sin^2 \vartheta - 1), & Y_{0,0,0}^{2,2} &= -\frac{1}{2} \cos^2 \vartheta \cos 2\vartheta_2. \end{aligned} \quad (\text{D.51})$$

These harmonics of the s^J scalar field in (5.91) correspond to the gauge theory operators $\text{Tr}(\Phi_3 + i\Phi_4)^2$, $\text{Tr}(\Phi_3 - i\Phi_4)^2$, $\text{Tr}(3(\Phi_1^2 + \Phi_2^2) - 1)$, and $\text{Tr}(\Phi_3^2 + \Phi_4^2 - \Phi_5^2 - \Phi_6^2)$ respectively. The spherical harmonics corresponding to the operators $\text{Tr}(\Phi_3 \pm i\Phi_4)^J$ for general J are

$$Y_{0,\pm J,0}^{J,J} = 2^{-J/2} \cos^J \vartheta \sin^J \vartheta_2 e^{\pm iJ\varphi_2}. \quad (\text{D.52})$$

The 50 Y^3 are given by

$$\begin{aligned} &\{z_1 z_2 z_3, z_1^* z_2 z_3, z_1^* z_2^* z_3, \dots, z_1^* z_2^* z_3^*\}, \\ &\{z_1 z_2^2, z_1 z_3^2, z_1 z_2^* z_3^2, z_1 z_3^* z_2^2\} + \text{cyclic permutations} + \text{c.c.}, \\ &\{z_1^3, z_2^3, z_3^3\} + \text{c.c.}, \\ &\{z_1(|z_2|^2 - |z_3|^2), z_2(4|z_1|^2 - 1), z_3(4|z_1|^2 - 1)\} + \text{c.c.}, \\ &\{z_1(|z_2|^2 + |z_3|^2 - 1/2), z_2(2|z_3|^2 - |z_2|^2), z_3(2|z_2|^2 - |z_3|^2)\} + \text{c.c.} \end{aligned} \quad (\text{D.53})$$

E AdS_5 metric fluctuations

The action of $D_\mu D_\nu$ on a scalar field Φ is,

$$D_\mu D_\nu \Phi = \partial_\mu \partial_\nu \Phi - \Gamma_{\mu\nu}^\lambda \partial_\lambda \Phi. \quad (\text{E.54})$$

The Christoffel symbols for the AdS_5 geometry are (comparing to (5.87), here we use $r_1 = r$, $\phi_1 = \phi$, $x = r_2 \cos \phi_2$, $z = r_2 \sin \phi_2$)

$$\begin{aligned} \Gamma_{\phi_i \phi_i}^{r_i} &= -r_i, & \Gamma_{\phi_i \phi_i}^y &= \frac{r_i^2}{y}, & \Gamma_{\phi_i r_i}^{\phi_i} &= \frac{1}{r_i}, & \Gamma_{\phi_i y}^{\phi_i} &= -\frac{1}{y}, \\ \Gamma_{r_i r_i}^y &= \frac{1}{y}, & \Gamma_{y r_i}^{r_i} &= -\frac{1}{y}, & \Gamma_{yy}^y &= -\frac{1}{y}, \end{aligned} \quad (\text{E.55})$$

where $i = 1, 2$. The trace of $D_\mu D_\nu \Phi$ is given by

$$g^{\mu\nu} D_\mu D_\nu \Phi = \left(y^2 \partial_y^2 - 3y \partial_y + \sum_{i=1}^2 \left(y^2 \partial_{r_i}^2 + \frac{y^2}{r_i^2} \partial_{\phi_i}^2 + \frac{y^2}{r_i} \partial_{r_i} \right) \right) \Phi. \quad (\text{E.56})$$

¹The normalization used is $\int_{S^5} |Y|^2 = 2^{1-J} \pi^3 / ((J+1)(J+2))$.

F An intriguing Matching

Computing the exchange of the SUGRA modes dual to

$$\mathcal{O}_J = \frac{1}{\sqrt{J\lambda}} \text{Tr} (\Phi_3 + i\Phi_4)^J \quad (\text{F.57})$$

that are the fluctuations of the RR 5-form as well as the spacetime metric and taking the expansion about small latitude radii θ_0 and θ_1 (where the polar angle of the latitudes at the south pole is given by $\pi - \theta_1$) we obtain these results :

$$\begin{aligned} J=2 : \quad \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} &= \frac{\lambda}{8N^2} \left[\frac{\theta_0^3 \theta_1^3}{2^2} + \frac{\theta_0^3 \theta_1^7 + \theta_0^7 \theta_1^3}{5 \cdot 3 \cdot 2^6} + \frac{\theta_0^5 \theta_1^5}{2^6} + \frac{\theta_0^3 \theta_1^9 + \theta_0^9 \theta_1^3}{7 \cdot 3^3 \cdot 2^6} \right. \\ &\quad \left. + \frac{\theta_0^5 \theta_1^7 + \theta_0^7 \theta_1^5}{3 \cdot 2^7} + \frac{\theta_0^6 \theta_1^6}{5^2 \cdot 3} - \frac{\theta_0^7 \theta_1^6 + \theta_0^6 \theta_1^7}{5 \cdot 3 \cdot 2^3} + \mathcal{O}(\theta^{14}) \right], \\ J=3 : \quad \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} &= \frac{\lambda}{32N^2} \left[\frac{3\theta_0^4 \theta_1^4}{8} + \frac{\theta_0^4 \theta_1^6 + \theta_0^6 \theta_1^4}{2^5} + \frac{3(\theta_0^4 \theta_1^8 + \theta_0^8 \theta_1^4)}{5 \cdot 2^7} \right. \\ &\quad \left. + \frac{5\theta_0^6 \theta_1^6}{3 \cdot 2^6} + \frac{3^3 \theta_0^7 \theta_1^7}{7^2 \cdot 5^2} + \frac{(\theta_0^6 \theta_1^8 + \theta_0^8 \theta_1^6)}{5 \cdot 2^5} + \frac{23(\theta_0^4 \theta_1^{10} + \theta_0^{10} \theta_1^4)}{7 \cdot 5 \cdot 3^3 \cdot 2^7} - \frac{3^2 (\theta_0^7 \theta_1^8 + \theta_0^8 \theta_1^7)}{7 \cdot 5 \cdot 2^5} + \mathcal{O}(\theta^{16}) \right], \\ J=4 : \quad \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} &= \frac{\lambda}{256N^2} \left[\theta_0^5 \theta_1^5 + \frac{\theta_0^5 \theta_1^7 + \theta_0^7 \theta_1^5}{3 \cdot 2} + \frac{\theta_0^5 \theta_1^9 + \theta_0^9 \theta_1^5}{3^2 \cdot 2^2} + \frac{13\theta_0^7 \theta_1^7}{3^2 \cdot 2^4} \right. \\ &\quad \left. + \mathcal{O}(\theta^{16}) \right]. \end{aligned} \quad (\text{F.58})$$

The YM₂ result in the large λ limit is

$$\left. \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \right|_{QCD_2} = \frac{\lambda \sin \theta_0 \sin \theta_1}{4N^2} \sum_{J=1}^{\infty} J \tan^J \frac{\theta_0}{2} \tan^J \frac{\theta_1}{2}. \quad (\text{F.59})$$

Ignoring $J=1$, we may expand in θ order-by-order in J :

$$\begin{aligned}
J=2 : \quad \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \Big|_{QCD_2} &= \frac{\lambda}{8N^2} \left[\frac{\theta_0^3 \theta_1^3}{2^2} + \frac{\theta_0^3 \theta_1^7 + \theta_0^7 \theta_1^3}{5 \cdot 3 \cdot 2^6} + \frac{\theta_0^3 \theta_1^9 + \theta_0^9 \theta_1^3}{7 \cdot 3^3 \cdot 2^6} + \mathcal{O}(\theta^{14}) \right], \\
J=3 : \quad \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \Big|_{QCD_2} &= \frac{\lambda}{32N^2} \left[\frac{3 \theta_0^4 \theta_1^4}{8} + \frac{\theta_0^4 \theta_1^6 + \theta_0^6 \theta_1^4}{2^5} + \frac{3 (\theta_0^4 \theta_1^8 + \theta_0^8 \theta_1^4)}{5 \cdot 2^7} + \frac{\theta_0^6 \theta_1^6}{3 \cdot 2^7} \right. \\
&\quad \left. + \frac{(\theta_0^6 \theta_1^8 + \theta_0^8 \theta_1^6)}{5 \cdot 2^9} + \frac{23 (\theta_0^4 \theta_1^{10} + \theta_0^{10} \theta_1^4)}{7 \cdot 5 \cdot 3^3 \cdot 2^7} + \mathcal{O}(\theta^{16}) \right], \\
J=4 : \quad \frac{\langle W(x) W(\bar{x}) \rangle}{\langle W(x) \rangle \langle W(\bar{x}) \rangle} \Big|_{QCD_2} &= \frac{\lambda}{256N^2} \left[\theta_0^5 \theta_1^5 + \frac{\theta_0^5 \theta_1^7 + \theta_0^7 \theta_1^5}{3 \cdot 2} + \frac{\theta_0^5 \theta_1^9 + \theta_0^9 \theta_1^5}{3^2 \cdot 2^2} + \frac{\theta_0^7 \theta_1^7}{3^2 \cdot 2^2} \right. \\
&\quad \left. + \mathcal{O}(\theta^{16}) \right].
\end{aligned} \tag{F.60}$$

There is a remarkable matching of highly non-trivial terms between these two calculations! The difference between the two calculations sets-in quite late

$$\begin{aligned}
(\text{SUGRA} - \text{QCD}_2)_{J=2} &= \frac{\lambda}{8N^2} \left[\frac{\theta_0^5 \theta_1^5}{2^6} + \frac{\theta_0^5 \theta_1^7 + \theta_0^7 \theta_1^5}{3 \cdot 2^7} + \frac{\theta_0^6 \theta_1^6}{5^2 \cdot 3} - \frac{\theta_0^7 \theta_1^6 + \theta_0^6 \theta_1^7}{5 \cdot 3 \cdot 2^3} + \mathcal{O}(\theta^{14}) \right], \\
(\text{SUGRA} - \text{QCD}_2)_{J=3} &= \frac{\lambda}{32N^2} \left[\frac{3 \theta_0^6 \theta_1^6}{2^7} + \frac{3 (\theta_0^6 \theta_1^8 + \theta_0^8 \theta_1^6)}{2^9} + \frac{3^3 \theta_0^7 \theta_1^7}{7^2 \cdot 5^2} \right. \\
&\quad \left. - \frac{3^2 (\theta_0^7 \theta_1^8 + \theta_0^8 \theta_1^7)}{7 \cdot 5 \cdot 2^5} + \mathcal{O}(\theta^{16}) \right], \\
(\text{SUGRA} - \text{QCD}_2)_{J=4} &= \frac{\lambda}{256N^2} \left[\frac{\theta_0^7 \theta_1^7}{2^4} + \mathcal{O}(\theta^{16}) \right].
\end{aligned} \tag{F.61}$$

Although we have considered values of J up to $J = 4$, we expect a similar pattern for arbitrary J .

G Divergences cancelation

Let us start to analyze the *self-energy diagram*. Using dimensional regularization ($d = 2\omega$), the one-loop correction to the gauge and scalar propagator in Feynman gauge is given by

$$S_{\text{self-energy}} = -g^4(N^2 - 1) \frac{\Gamma^2(\omega - 1)}{2^7 \pi^{2\omega} (2 - \omega)(2\omega - 3)} \oint d\tau_1 d\tau_2 \frac{(\omega_{x_1} \cdot \omega_{x_2} - \dot{x}_1 \cdot \dot{x}_2)}{[(x_1 - x_2)^2]^{2\omega - 3}} \tag{G.62}$$

The coefficient exhibits a pole in $\omega = 2$ and thus this contribution is divergent. For the so called *spider diagrams*, namely the perturbative contributions coming from the gauge vertex A^3 and the scalar-gauge vertex $\phi^2 A$, we have

$$S_{spider} = \frac{g^4(N^2 - 1)}{4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) (\omega_{x_1} \cdot \omega_{x_2} - \dot{x}_1 \cdot \dot{x}_2) \dot{x}_2^\mu \frac{\partial \mathcal{I}_1(x_3 - x_1, x_2 - x_1)}{\partial x_3^\mu}, \quad (\text{G.63})$$

where the symbol $\epsilon(t_1, t_2, t_3) = 1$ when $t_1 > t_2 > t_3$ and it is antysymmetric under any permutation of t_i ($\mathcal{I}_1(x, y)$ is defined in *appendix C*). Rearranging the expression for S_{spider} with the help of this trivial identity:

$$0 = \frac{g^4(N^2 - 1)}{4} \oint dt_1 dt_2 dt_3 \frac{d}{dt_2} \left[\epsilon(t_1, t_2, t_3) ((\omega_{x_1} \cdot \omega_{x_2} - \dot{x}_1 \cdot \dot{x}_2)) \mathcal{I}_2(x_3 - x_2, x_1 - x_2) \right]. \quad (\text{G.64})$$

(where the function \mathcal{I}_2 is defined in *appendix C*), one can note that the sum of the divergent part in the *spider diagram* plus the *self energy* contribution gives a finite terms that together with the finite part of the *spider* can be recast in a compact useful form (free of divergencies) :

$$\begin{aligned} \mathcal{I}_{\text{tot}} = & \frac{g^4(N^2 - 1)}{128\pi^4} \oint dt_1 dt_2 dt_3 \epsilon(t_1, t_2, t_3) \frac{(\omega_{x_3} \cdot \omega_{x_1} - \dot{x}_1 \cdot \dot{x}_3)}{(x_3 - x_1)^2} \times \\ & \times \frac{(x_3 - x_2) \cdot \dot{x}_2}{(x_3 - x_2)^2} \log \left(\frac{(x_2 - x_1)^2}{(x_3 - x_1)^2} \right). \end{aligned} \quad (\text{G.65})$$

H Interaction Diagrams

We have to evaluated numerically the function \mathcal{I}_{tot} for the case *B* (depicted in fig.(4.7)) in which two of the three legs of the vertex are attached to the same edge of the circuits and for the case *C* (depicted in fig.(4.8)) in which all the legs are attached to a different edge.

Interactions (*B*) diagrams : We begin with the diagrams depicted in figure IV, V and VI (chapter 4) that in a compact formula can be written as

$$\begin{aligned} \mathcal{I}_{IV,V,VI} = g^4(N^2 - 1) & \left[\left(\int_0^\delta dt_1 \int_0^\theta dt_2 dt_3 \text{sgn}(t_2 - t_3) V_1 \right) + \right. \\ & \left. + \left(\int_0^\delta dt_2 \int_0^\theta dt_3 dt_1 \text{sgn}(t_3 - t_1) V_2 \right) + \left(\int_0^\delta dt_3 \int_0^\theta dt_1 dt_2 \text{sgn}(t_1 - t_2) V_3 \right) \right] \end{aligned} \quad (\text{H.66})$$

where

$$\begin{aligned} V_1 &= \frac{\sin(t_1) \sinh(\theta) \sinh(t_2 - t_3) (\cosh(t_3) - \cosh(\theta)) \text{Log} \left(\frac{\cos(t_1) \sinh(t_2) \sinh(\theta) - \cosh(t_2) \cosh(\theta) + 1}{\cos(t_1) \sinh(t_3) \sinh(\theta) - \cosh(t_3) \cosh(\theta) + 1} \right)}{512\pi^4 (\cosh(t_2 - t_3) - 1) (-\cos(t_1) \sinh(t_3) \sinh(\theta) + \cosh(t_3) \cosh(\theta) - 1)} \\ V_2 &= \frac{\sin(t_2) \sinh(t_3) \sinh(\theta) \sinh^2 \left(\frac{t_3 - t_1}{2} \right) \text{Log} \left(\frac{\sinh(t_1) \cos(t_2) \sinh(\theta) - \cosh(t_1) \cosh(\theta) + 1}{1 - \cosh(t_1 - t_3)} \right)}{256\pi^4 (\cosh(t_1 - t_3) - 1) (\cos(t_2) \sinh(t_3) \sinh(\theta) - \cosh(t_3) \cosh(\theta) + 1)} \\ V_3 &= \frac{\sin(t_3) \sinh(\theta) (\cosh(t_1) - \cosh(\theta)) (\cos(t_3) \sinh(\theta) \cosh(t_2 - \theta) + \cosh(\theta) \sinh(t_2 - \theta))}{512\pi^4 (\sinh(t_1) \cos(t_3) \sinh(\theta) - \cosh(t_1) \cosh(\theta) + 1) (\cos(t_3) \sinh(\theta) \sinh(t_2 - \theta) + \cosh(\theta) \cosh(t_2 - \theta) - 1)} \times \\ & \quad \text{Log} \left(\frac{1 - \sinh(t_1) \sinh(t_2 - \theta) - \cosh(t_1) \cosh(t_2 - \theta)}{\sinh(t_1) \cos(t_3) \sinh(\theta) - \cosh(t_1) \cosh(\theta) + 1} \right). \end{aligned} \quad (\text{H.67})$$

These diagrams are identically to the diagrams (XVI, XVII and XVIII) (chapter 4) since one cannot distinguish the two rays of the cusp and thus we have to consider two times the contribution of $\mathcal{I}_{IV,V,VI}$. Then we pass to study the diagrams indicated with the number (VII, VIII, IX) (chapter 4). Their value is given by the expression

$$\begin{aligned} \mathcal{I}_{VII,VIII,IX} = g^4(N^2 - 1) & \left[\left(\int_{-\theta}^0 dt_1 \int_0^\theta dt_2 dt_3 \text{sgn}(t_2 - t_3) Y_1 \right) + \right. \\ & \left. + \left(\int_{-\theta}^0 dt_2 \int_0^\theta dt_1 dt_3 \text{sgn}(t_3 - t_1) Y_2 \right) + \left(\int_{-\theta}^0 dt_3 \int_0^\theta dt_1 dt_2 \text{sgn}(t_1 - t_2) Y_3 \right) \right] \end{aligned} \quad (\text{H.68})$$

where

$$\begin{aligned} Y_1 &= - \frac{\sinh(t_2 - t_3) (\cos(\delta) (\cosh(t_1) \cosh(t_3) - 1) + \sinh(t_1) \sinh(t_3)) \text{Log} \left(\frac{\sinh(t_1) \sinh(t_2) \cos(\delta) + \cosh(t_1) \cosh(t_2) - 1}{\sinh(t_1) \sinh(t_3) \cos(\delta) + \cosh(t_1) \cosh(t_3) - 1} \right)}{512\pi^4 (\cosh(t_2 - t_3) - 1) (\sinh(t_1) \sinh(t_3) \cos(\delta) + \cosh(t_1) \cosh(t_3) - 1)} \\ Y_2 &= \frac{(\cosh(t_2) \sinh(t_3) \cos(\delta) + \sinh(t_2) \cosh(t_3)) \text{Log} \left(\frac{\sinh(t_1) \sinh(t_2) \cos(\delta) + \cosh(t_1) \cosh(t_2) - 1}{\cosh(t_1 - t_3) - 1} \right)}{512\pi^4 (\sinh(t_2) \sinh(t_3) \cos(\delta) + \cosh(t_2) \cosh(t_3) - 1)} \\ Y_3 &= - \frac{(\cos(\delta) (\cosh(t_1) \cosh(t_3) - 1) + \sinh(t_1) \sinh(t_3)) (\cosh(t_2) \sinh(t_3) \cos(\delta) + \sinh(t_2) \cosh(t_3))}{512\pi^4 (\sinh(t_1) \sinh(t_3) \cos(\delta) + \cosh(t_1) \cosh(t_3) - 1) (\sinh(t_2) \sinh(t_3) \cos(\delta) + \cosh(t_2) \cosh(t_3) - 1)} \times \\ & \quad \text{Log} \left(\frac{\cosh(t_1 - t_2) - 1}{\sinh(t_1) \sinh(t_3) \cos(\delta) + \cosh(t_1) \cosh(t_3) - 1} \right). \end{aligned} \quad (\text{H.69})$$

Since this contribution and the one coming from diagrams (XIX, XX, XXI) is the same and similarly the diagrams (XIII, XIV, XV) and (X, XI, XII) are equal, the value of the last three (*B*) type diagrams to consider is given by

$$\mathcal{I}_{X, XI, XII} = g^4 (N^2 - 1) \left[\left(\int_{-\theta}^0 dt_1 \int_{\pi-\delta}^{\pi} dt_2 dt_3 \text{sgn}(t_2 - t_3) W_1 \right) + \right. \\ \left. \left(\int_{-\theta}^0 dt_2 \int_{\pi-\delta}^{\pi} dt_1 dt_3 \text{sgn}(t_3 - t_1) W_2 \right) + \left(\int_{-\theta}^0 dt_3 \int_{\pi-\delta}^{\pi} dt_1 dt_2 \text{sgn}(t_1 - t_2) W_3 \right) \right] \quad (\text{H.70})$$

where

$$W_1 = \frac{\sin(t_3) \sinh(\theta) (\cosh(t_1) - \cosh(\theta)) \sin(t_2 - t_3) \text{Log} \left(\frac{\sinh(t_1) \cos(t_2) \sinh(\theta) - \cosh(t_1) \cosh(\theta) + 1}{\sinh(t_1) \cos(t_3) \sinh(\theta) - \cosh(t_1) \cosh(\theta) + 1} \right)}{512\pi^4 (\cos(t_2 - t_3) - 1) (-\sinh(t_1) \cos(t_3) \sinh(\theta) + \cosh(t_1) \cosh(\theta) - 1)}$$

$$W_2 = \frac{\sinh^2(\theta) \sin^2\left(\frac{t_1 - t_3}{2}\right) (\sinh(t_2) \cosh(\theta) - \cosh(t_2) \cos(t_3) \sinh(\theta))}{256\pi^4 (\cos(t_1 - t_3) - 1) (\sinh(t_2) \cos(t_3) \sinh(\theta) - \cosh(t_2) \cosh(\theta) + 1)} \times \\ \text{Log} \left(\frac{4 \cos(t_1) \sinh(t_2) \text{csch}(\theta) - 4 \cosh(t_2) \coth(\theta) \text{csch}(\theta) + \coth^2(\theta) + 3 \text{csch}^2(\theta) - 1}{4(\cos(t_1 - t_3) - 1)} \right)$$

$$W_3 = \frac{\sin(t_1) \sin(t_2) \sinh(t_3) \sinh^2(\theta) (\cosh(\theta) - \cosh(t_3)) \text{Log} \left(\frac{\sinh^2(\theta) (\cos(t_1 - t_2) - 1)}{\cos(t_1) \sinh(t_3) \sinh(\theta) - \cosh(t_3) \cosh(\theta) + 1} \right)}{512\pi^4 (-\cos(t_1) \sinh(t_3) \sinh(\theta) + \cosh(t_3) \cosh(\theta) - 1) (-\cos(t_2) \sinh(t_3) \sinh(\theta) + \cosh(t_3) \cosh(\theta) - 1)}. \quad (\text{H.71})$$

Interactions (C) diagrams : Now we pass to analyze the (C) type diagrams. We immediately note that the six diagrams can be divided in two equivalent groups. The total result is thus given by two times this contribution

$$\mathcal{I}_{XXII, XXVII} = g^4 (N^2 - 1) \left[\int_{\pi-\delta}^{\pi} dt_1 \int_0^{\theta} dt_2 \int_{-\theta}^0 dt_3 C_1 + \int_{\pi-\delta}^{\pi} dt_2 \int_0^{\theta} dt_3 \int_{-\theta}^0 dt_1 C_2 + \right. \\ \left. \int_{\pi-\delta}^{\pi} dt_3 \int_0^{\theta} dt_1 \int_{-\theta}^0 dt_2 C_3 \right] \quad (\text{H.72})$$

where

$$C_1 = -\frac{\sin(t_1) \sinh(\theta) (\cosh(t_3) - \cosh(\theta)) (\cosh(t_2) \sinh(t_3) \cos(\delta) + \sinh(t_2) \cosh(t_3))}{512\pi^4 (-\cos(t_1) \sinh(t_3) \sinh(\theta) + \cosh(t_3) \cosh(\theta) - 1) (\sinh(t_2) \sinh(t_3) \cos(\delta) + \cosh(t_2) \cosh(t_3) - 1)} \times \\ \text{Log} \left(\frac{\sinh(t_2) \sinh(\theta) \cos(t_1 + \delta) + \cosh(t_2) \cosh(\theta) - 1}{-\cos(t_1) \sinh(t_3) \sinh(\theta) + \cosh(t_3) \cosh(\theta) - 1} \right)$$

$$C_2 = \frac{\sinh(t_3) \sinh(\theta) \sin(t_2 + \delta) (\cos(\delta) (\cosh(t_1) \cosh(t_3) - 1) + \sinh(t_1) \sinh(t_3))}{512\pi^4 (\sinh(t_1) \sinh(t_3) \cos(\delta) + \cosh(t_1) \cosh(t_3) - 1) (\sinh(t_3) \sinh(\theta) \cos(t_2 + \delta) + \cosh(t_3) \cosh(\theta) - 1)} \times \\ \text{Log} \left(\frac{-\sinh(t_1) \cos(t_2) \sinh(\theta) + \cosh(t_1) \cosh(\theta) - 1}{\sinh(t_1) \sinh(t_3) \cos(\delta) + \cosh(t_1) \cosh(t_3) - 1} \right)$$

$$C_3 = \frac{\sinh(\theta) (\cosh(t_1) - \cosh(\theta)) \sin(t_3 + \delta) (\sinh(t_2) \cosh(\theta) - \cosh(t_2) \cos(t_3) \sinh(\theta))}{512\pi^4 (-\sinh(t_2) \cos(t_3) \sinh(\theta) + \cosh(t_2) \cosh(\theta) - 1) (\sinh(t_1) \sinh(\theta) \cos(t_3 + \delta) + \cosh(t_1) \cosh(\theta) - 1)} \times \\ \text{Log} \left(\frac{\sinh(t_1) \sinh(t_2) \cos(\delta) + \cosh(t_1) \cosh(t_2) - 1}{\sinh(t_1) \sinh(\theta) \cos(t_3 + \delta) + \cosh(t_1) \cosh(\theta) - 1} \right). \quad (\text{H.73})$$

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